Multilinear Algebraic Branching Programs and the Min-Partition Rank Method

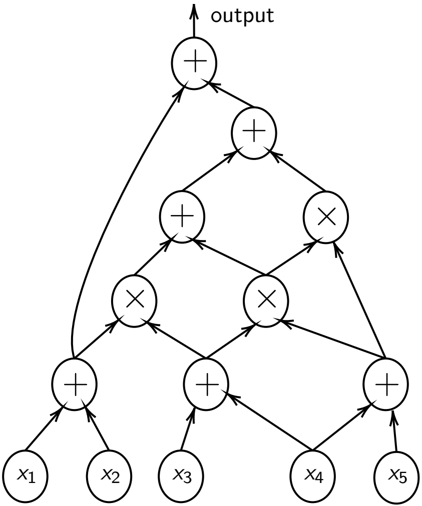
Théo Fabris (U. of Copenhagen)
Nutan Limaye (I.T.U. of Copenhagen)
Srikanth Srinivasan (U. of Copenhagen)
Amir Yehudayoff (U. of Copenhagen)

CCC, August 6, 2026

Summary

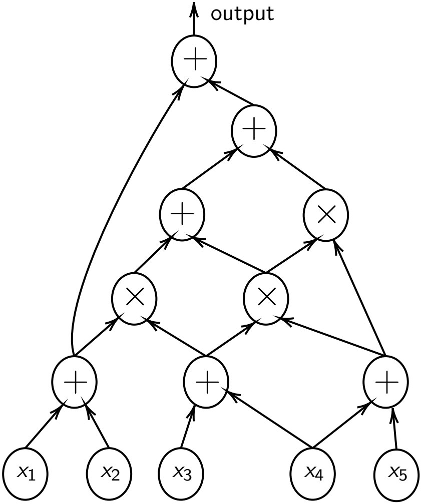
Formulas, ABPs, and Circuits Hierarchy

Formulas, ABPs, and Circuits Hierarchy

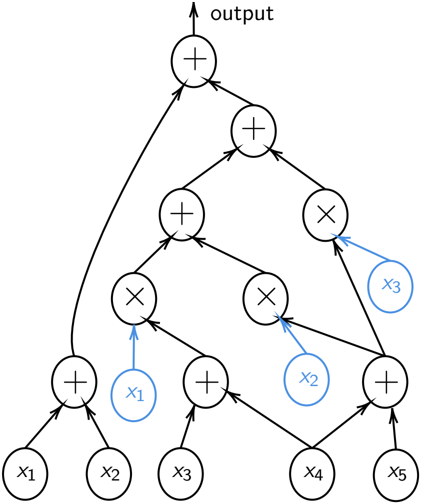


Circuits = general DAGs

Formulas, ABPs, and Circuits Hierarchy

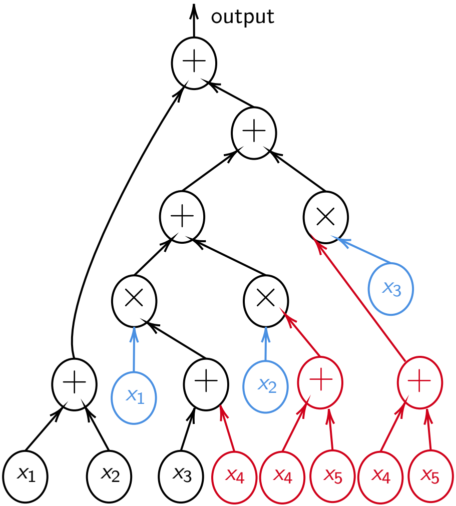


Circuits = general DAGs

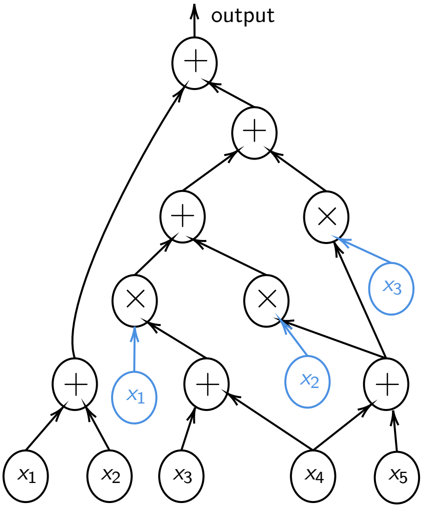


ABPs = skew circuits

Formulas, ABPs, and Circuits Hierarchy

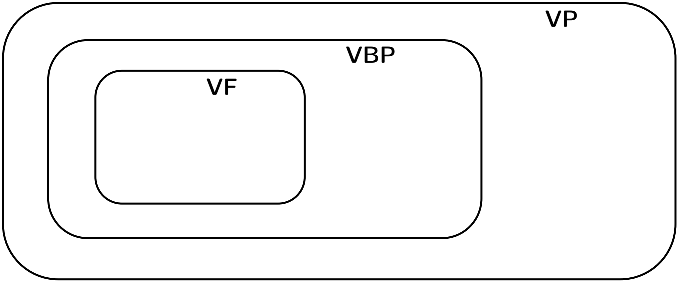


Formulas = tree circuits



ABPs = skew circuits

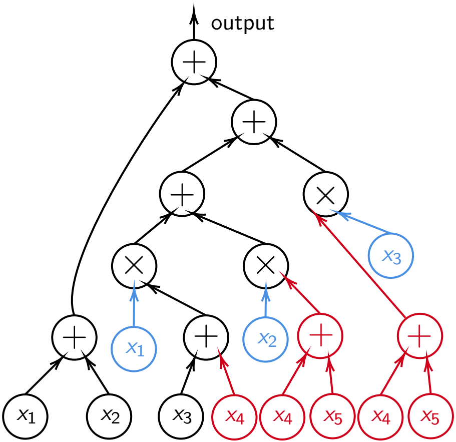
Formulas, ABPs, and Circuits Hierarchy



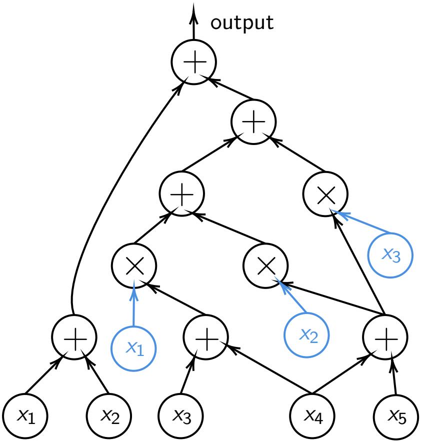
Open question: Prove lower bounds for **VF**, **VBP** and **VP**.

(Syntactically) multilinear **VF**, **VBP**, and **VP**

(Syntactically) multilinear **VF**, **VBP**, and **VP**



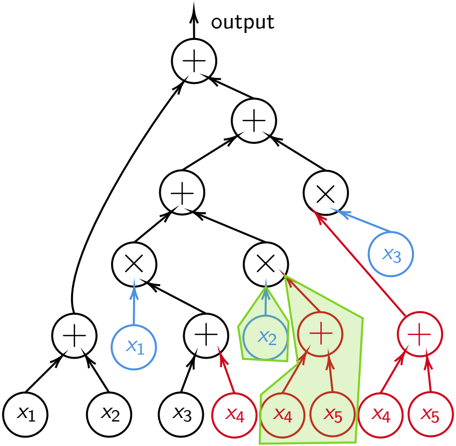
Formulas = tree circuits



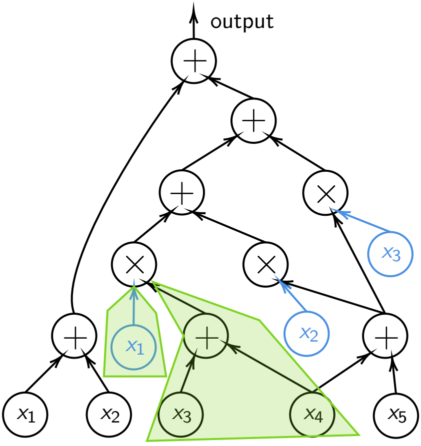
ABPs = skew circuits

(Syntactically) multilinear **VF**, **VBP**, and **VP**

Multilinear restriction: every multiplication gate has all its children's subcircuits using disjoint set of variables.

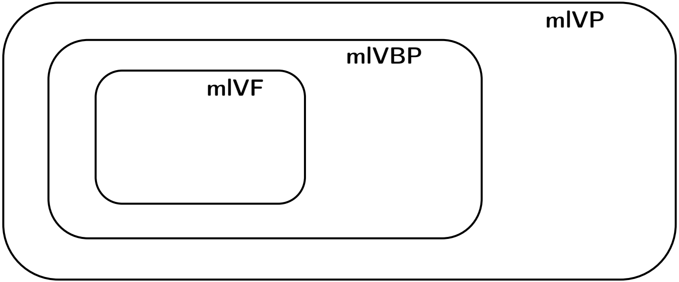


Formulas = tree circuits



ABPs = skew circuits

(Syntactically) multilinear **VF**, **VBP**, and **VP**



Open questions: Prove lower bounds for **mlVBP** and **mlVP**.

Raz's min-partition rank method

Raz's min-partition rank method

$P \in \mathbb{F}[x_1, \cdots, x_n]$ is **full-rank** if, $\forall$ balanced partition $f : [n] \to \{\pm 1\}$,

$\mathrm{rk}(M_{f,P}) =$ **full rank**

Raz's min-partition rank method

$P \in \mathbb{F}[x_1, \cdots, x_n]$ is **full-rank** if, $\forall$ balanced partition $f : [n] \to \{\pm 1\}$,

$$\mathrm{rk}(M_{f,P}) = 2^{n/2}, \text{ where } M_{f,P}(m_1, m_2) := \mathrm{coeff}(P, m_1 \cdot m_2)$$

$\forall$ monomials $m_1 \in \mathbb{F}[x_i | f(i) = 1]$, $m_2 \in \mathbb{F}[x_i | f(i) = -1]$.

Raz's min-partition rank method

$$P := 1 + y_1z_1 + y_2z_2 + y_3z_3 + y_1y_2z_1z_2 + y_1y_3z_1z_3 + y_2y_3z_2z_3 + y_1y_2y_3z_1z_2z_3$$

$$f(y_i) := 1, f(z_i) := -1$$

$$M_{f,P} = \begin{array}{cc} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} & \begin{array}{l} 1 \\ y_1 \\ y_2 \\ y_3 \\ y_1y_2 \\ y_1y_3 \\ y_2y_3 \\ y_1y_2y_3 \end{array} \\ \begin{array}{cccccccc} 1 & z_1 & z_2 & z_3 & & \cdots & & \end{array} & \end{array}$$

$P \in \mathbb{F}[x_1, \cdots, x_n]$ is **full-rank** if, $\forall$ balanced partition $f : [n] \to \{\pm 1\}$,

$$\mathrm{rk}(M_{f,P}) = 2^{n/2}, \text{ where } M_{f,P}(m_1, m_2) := \mathrm{coeff}(P, m_1 \cdot m_2)$$

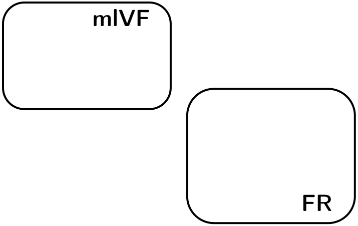
$\forall$ monomials $m_1 \in \mathbb{F}[x_i | f(i) = 1]$, $m_2 \in \mathbb{F}[x_i | f(i) = -1]$.

Raz's min-partition rank method

$P \in \mathbb{F}[x_1, \cdots, x_n]$ is **full-rank** if, $\forall$ balanced partition $f : [n] \to \{\pm 1\}$,

$$\mathrm{rk}(M_{f,P}) = 2^{n/2}, \text{ where } M_{f,P}(m_1, m_2) := \mathrm{coeff}(P, m_1 \cdot m_2)$$

$\forall$ monomials $m_1 \in \mathbb{F}[x_i | f(i) = 1], m_2 \in \mathbb{F}[x_i | f(i) = -1]$.



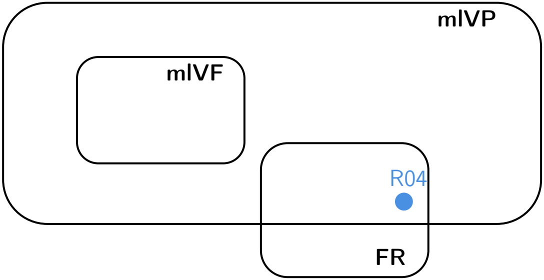
FR $:= \{$full rank polynomials$\}$

Raz's min-partition rank method

$P \in \mathbb{F}[x_1, \cdots, x_n]$ is **full-rank** if, $\forall$ balanced partition $f : [n] \to \{\pm 1\}$,

$$\mathrm{rk}(M_{f,P}) = 2^{n/2}, \text{ where } M_{f,P}(m_1, m_2) := \mathrm{coeff}(P, m_1 \cdot m_2)$$

$\forall$ monomials $m_1 \in \mathbb{F}[x_i | f(i) = 1], m_2 \in \mathbb{F}[x_i | f(i) = -1]$.



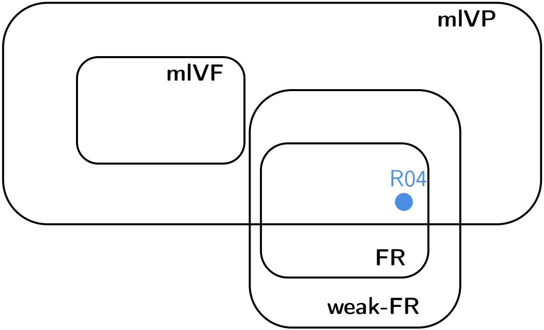
FR $:= \{$full rank polynomials$\}$

Raz's min-partition rank method

$P \in \mathbb{F}[x_1, \cdots, x_n]$ is **full-rank** if, $\forall$ balanced partition $f : [n] \to \{\pm 1\}$,

$$\mathrm{rk}(M_{f,P}) = 2^{n/2}, \text{ where } M_{f,P}(m_1, m_2) := \mathrm{coeff}(P, m_1 \cdot m_2)$$

$\forall$ monomials $m_1 \in \mathbb{F}[x_i | f(i) = 1]$, $m_2 \in \mathbb{F}[x_i | f(i) = -1]$.



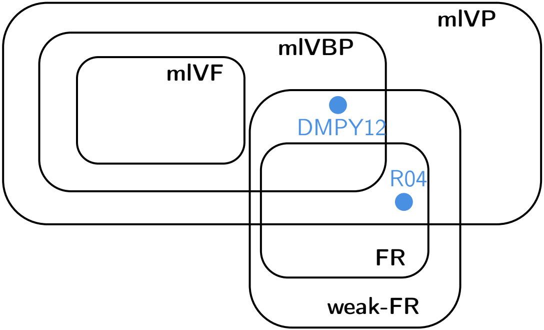
FR := {full rank polynomials}

Raz's min-partition rank method

$P \in \mathbb{F}[x_1, \cdots, x_n]$ is **full-rank** if, $\forall$ balanced partition $f : [n] \to \{\pm 1\}$,

$$\mathrm{rk}(M_{f,P}) = 2^{n/2}, \text{ where } M_{f,P}(m_1, m_2) := \mathrm{coeff}(P, m_1 \cdot m_2)$$

$\forall$ monomials $m_1 \in \mathbb{F}[x_i | f(i) = 1], m_2 \in \mathbb{F}[x_i | f(i) = -1]$.



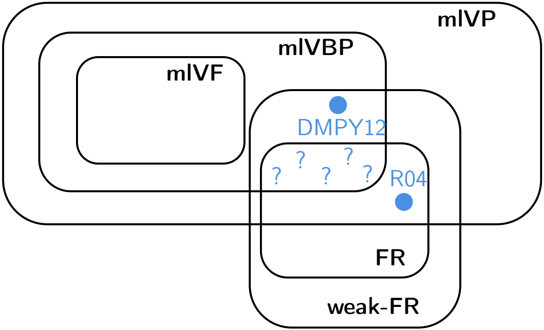
FR $:= \{$full rank polynomials$\}$

Raz's min-partition rank method

$P \in \mathbb{F}[x_1, \cdots, x_n]$ is **full-rank** if, $\forall$ balanced partition $f : [n] \to \{\pm 1\}$,

$$\mathrm{rk}(M_{f,P}) = 2^{n/2}, \text{ where } M_{f,P}(m_1, m_2) := \mathrm{coeff}(P, m_1 \cdot m_2)$$

$\forall$ monomials $m_1 \in \mathbb{F}[x_i | f(i) = 1]$, $m_2 \in \mathbb{F}[x_i | f(i) = -1]$.



FR := {full rank polynomials}

Goal:
Understand FR ∩ mlVBP

Min-partition rank as mlABP lower bound criterion?

Min-partition rank as mlABP lower bound criterion?

Theorem 1

The min. size of a mlABP computing a full rank polynomial

Min-partition rank as mlABP lower bound criterion?

Theorem 1

The min. size of a mlABP computing a full rank polynomial is ***almost characterized*** *by the min. size of certain combinatorial set systems.*

Min-partition rank as mlABP lower bound criterion?

Theorem 1

The min. size of a mlABP computing a full rank polynomial is ***almost characterized*** *by the min. size of certain combinatorial set systems.*

Using our combinatorial framework, we obtain:

Min-partition rank as mlABP lower bound criterion?

Theorem 1

The min. size of a mlABP computing a full rank polynomial is ***almost characterized*** *by the min. size of certain combinatorial set systems.*

Using our combinatorial framework, we obtain:

- Improved **exponential lower bounds** for Σ_π**mlABP** (CKSS'24).

Min-partition rank as mlABP lower bound criterion?

Theorem 1

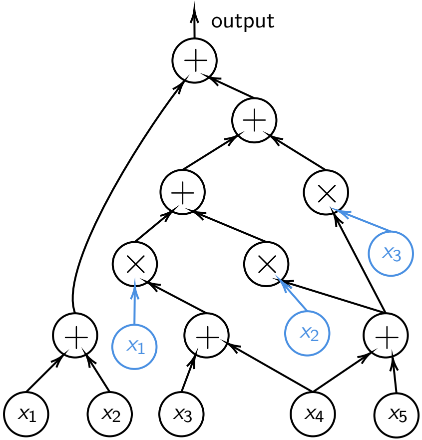
The min. size of a mlABP computing a full rank polynomial is ***almost characterized*** *by the min. size of certain combinatorial set systems.*

Using our combinatorial framework, we obtain:

- Improved **exponential lower bounds** for Σ_π**mlABP** (CKSS'24).
- **Superpolynomial separations** between multilinear formulas and mlABPs, but weaker than the known optimal separation (DMPY'12).

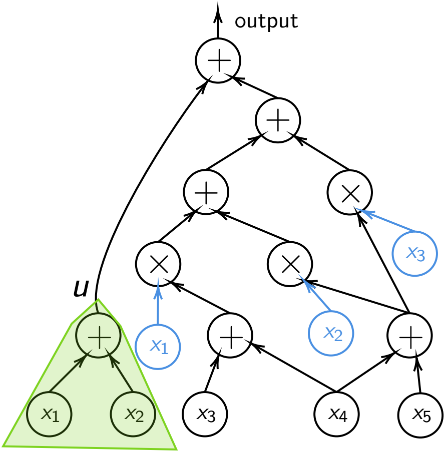
Multilinear algebraic branching programs and set systems

Multilinear algebraic branching programs and set systems



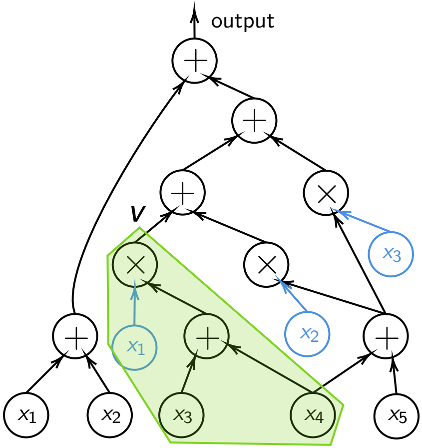
ABPs = skew circuits

Multilinear algebraic branching programs and set systems



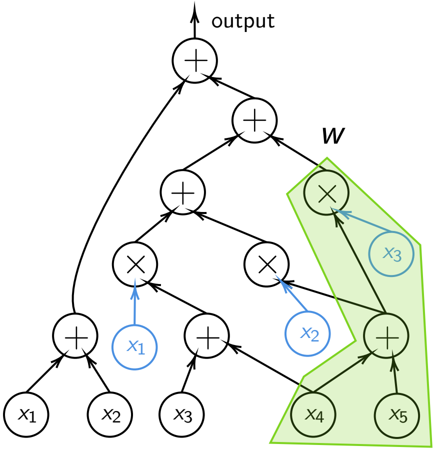
ABPs = skew circuits

Multilinear algebraic branching programs and set systems



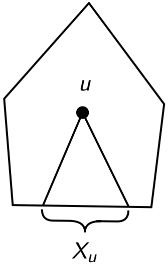
ABPs = skew circuits

Multilinear algebraic branching programs and set systems

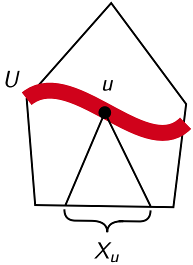


ABPs = skew circuits

Multilinear algebraic branching programs and set systems

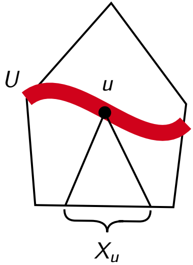


Multilinear algebraic branching programs and set systems



Multilinear algebraic branching programs and set systems

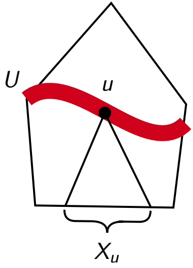
$$P = \sum_{u \in U} \overset{\overset{\mathbb{F}[X_u]}{\rotatebox{90}{$\in$}}}{A_u} \cdot \overset{\overset{\mathbb{F}[Y_u]}{\rotatebox{90}{$\in$}}}{B_u}$$



$Y_u := [n] \setminus X_u$

Multilinear algebraic branching programs and set systems

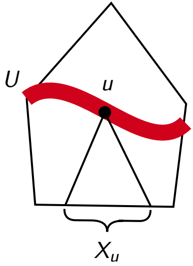
$$P = \sum_{u \in U} \overset{\overset{\mathbb{F}[X_u]}{\rotatebox{90}{$\in$}}}{A_u} \cdot \overset{\overset{\mathbb{F}[Y_u]}{\rotatebox{90}{$\in$}}}{B_u}$$



$Y_u := [n] \setminus X_u$

$$\boxed{M_{f,A\cdot B}}$$

Multilinear algebraic branching programs and set systems

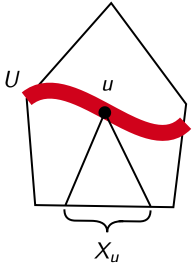


$$P = \sum_{u \in U} \underset{\mathbb{F}[X_u]}{A_u} \cdot \underset{\mathbb{F}[Y_u]}{B_u}, \qquad A_u \in \mathbb{F}[X_u], \quad B_u \in \mathbb{F}[Y_u]$$

$$\boxed{M_{f,A\cdot B}} = \boxed{M_{g,A}} \otimes \boxed{M_{h,B}}$$

$Y_u := [n] \setminus X_u \qquad g := f\restriction_{X_u}, h := f\restriction_{Y_u}$

Multilinear algebraic branching programs and set systems



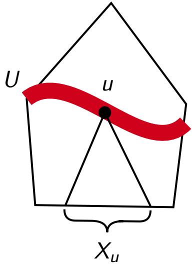
$$P = \sum_{u \in U} \overset{\mathbb{F}[X_u]}{\overset{\cup}{A_u}} \cdot \overset{\mathbb{F}[Y_u]}{\overset{\cup}{B_u}}$$

$$R := f^{-1}(1), C := f^{-1}(-1)$$

$$\begin{array}{cc} R & \boxed{M_{f,A\cdot B}} \\ & C \end{array} = \boxed{M_{g,A}} \otimes \boxed{M_{h,B}}$$

$Y_u := [n] \setminus X_u$ $\qquad g := f\restriction_{X_u}, h := f\restriction_{Y_u}$

Multilinear algebraic branching programs and set systems



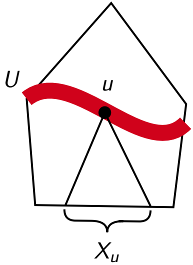
$$P = \sum_{u\in U} \overset{\mathbb{F}[X_u]}{\overset{\cup}{A_u}} \cdot \overset{\mathbb{F}[Y_u]}{\overset{\cup}{B_u}}$$

$$R := f^{-1}(1),\ C := f^{-1}(-1)$$

$$R\ \underset{C}{\boxed{M_{f,A\cdot B}}} = R\cap X_u\ \underset{C\cap X_u}{\boxed{M_{g,A}}} \otimes \underset{C\cap Y_u}{\boxed{M_{h,B}}}\ R\cap Y_u$$

$Y_u := [n] \setminus X_u$ $\qquad$ $g := f\restriction_{X_u},\ h := f\restriction_{Y_u}$

Multilinear algebraic branching programs and set systems



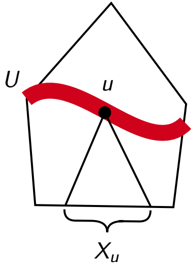
$$P = \sum_{u \in U} \overset{\mathbb{F}[X_u]}{\overset{\in}{A_u}} \cdot \overset{\mathbb{F}[Y_u]}{\overset{\in}{B_u}} \quad \Rightarrow \mathrm{rk}(M_{f,P}) \leq \sum_{u \in U} 2^{n/2} \cdot 2^{-||X_u|-|Y_u||}$$

$$R := f^{-1}(1), C := f^{-1}(-1)$$

$$R \underset{C}{\boxed{M_{f,A\cdot B}}} = R \cap X_u \underset{C \cap X_u}{\boxed{M_{g,A}}} \otimes \underset{C \cap Y_u}{\boxed{M_{h,B}}} R \cap Y_u$$

$$Y_u := [n] \setminus X_u \qquad g := f\restriction_{X_u}, h := f\restriction_{Y_u}$$

Multilinear algebraic branching programs and set systems



$$P = \sum_{u \in U} \overset{\mathbb{F}[X_u]}{\overset{\rotatebox{90}{$\in$}}{A_u}} \cdot \overset{\mathbb{F}[Y_u]}{\overset{\rotatebox{90}{$\in$}}{B_u}} \quad \Rightarrow \mathrm{rk}(M_{f,P}) \leq \sum_{u \in U} 2^{n/2} \cdot 2^{-||X_u| - |Y_u||} \color{red}{\leq 2^{n/2} \cdot |U| \cdot 2^{-k} \text{ ????}}$$

$$R := f^{-1}(1), C := f^{-1}(-1)$$

$$\underset{C}{R \boxed{M_{f,A \cdot B}}} = \underset{C \cap X_u}{R \cap X_u \boxed{M_{g,A}}} \otimes \underset{C \cap Y_u}{\boxed{M_{h,B}} R \cap Y_u}$$

$$Y_u := [n] \setminus X_u \qquad g := f\restriction_{X_u}, h := f\restriction_{Y_u}$$

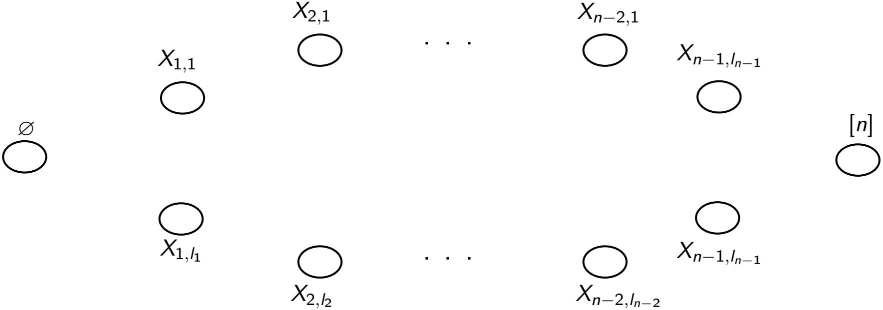
Approach:
Rank upper bounds for a **mlABP** P via properties of $\{ X_u \mid u \in P \}$?

Set systems and their chain-balance

Formal definitions: $\mathcal{X} \subseteq \mathcal{P}([n])$ set system over $[n]$.

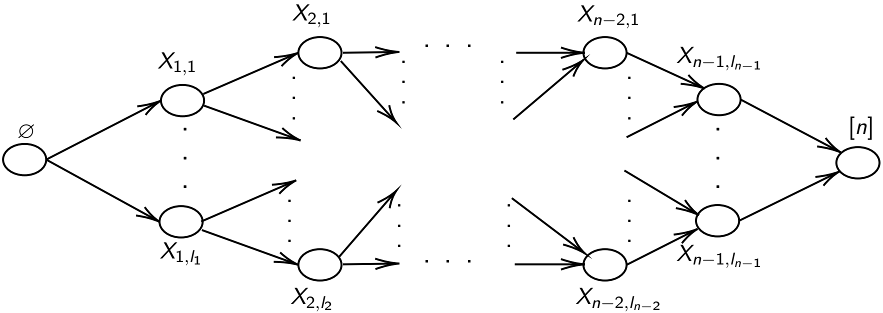
Set systems and their chain-balance

Formal definitions: $\mathcal{X} \subseteq \mathcal{P}([n])$ set system over $[n]$.



Set systems and their chain-balance

Formal definitions: $\mathcal{X} \subseteq \mathcal{P}([n])$ set system over $[n]$.



(U, V) is an arc $\iff$ $V = U \cup \{v\}$

Set systems and their chain-balance

Formal definitions: $\mathcal{X} \subseteq \mathcal{P}([n])$ set system over $[n]$.

- For a partition $f\colon [n] \to \{\pm 1\}$, we say that $\mathcal{X}$ is k-**chain-balanced w.r.t.** f if

Set systems and their chain-balance

Formal definitions: $\mathcal{X} \subseteq \mathcal{P}([n])$ set system over $[n]$.

- For a partition $f : [n] \to \{\pm 1\}$, we say that $\mathcal{X}$ is k-**chain-balanced w.r.t.** f if there is a chain

$$X_0 \subsetneq X_1 \subsetneq \ldots \subsetneq X_n$$

s.t. $|X_i| = |X_{i-1}| + 1$ and $X_0 = \varnothing$ and

$$|f(X_i)| := |\sum_{x \in X_i} f(x)| \leq k.$$

Set systems and their chain-balance

Formal definitions: $\mathcal{X} \subseteq \mathcal{P}([n])$ set system over $[n]$.

- For a partition $f\colon [n] \to \{\pm 1\}$, we say that $\mathcal{X}$ is k-**chain-balanced w.r.t.** f if there is a chain

  $$X_0 \subsetneq X_1 \subsetneq \ldots \subsetneq X_n$$

  s.t. $|X_i| = |X_{i-1}| + 1$ and $X_0 = \varnothing$ and

  $$|f(X_i)| := |\sum_{x \in X_i} f(x)| \leq k.$$

  If this is the case, we write $\mathrm{cbal}(\mathcal{X}, f) \leq k$.

Set systems and their chain-balance

Formal definitions: $\mathcal{X} \subseteq \mathcal{P}([n])$ set system over $[n]$.

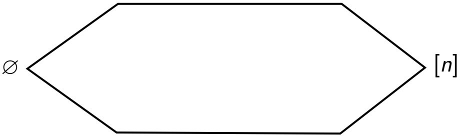
- For a partition $f\colon [n] \to \{\pm 1\}$, we say that $\mathcal{X}$ is k-**chain-balanced w.r.t.** f if there is a chain

$$X_0 \subsetneq X_1 \subsetneq \ldots \subsetneq X_n$$

  s.t. $|X_i| = |X_{i-1}| + 1$ and $X_0 = \varnothing$ and

$$|f(X_i)| := |\sum_{x \in X_i} f(x)| \leq k.$$

  If this is the case, we write $\mathrm{cbal}(\mathcal{X}, f) \leq k$.



Set systems and their chain-balance

Formal definitions: $\mathcal{X} \subseteq \mathcal{P}([n])$ set system over $[n]$.

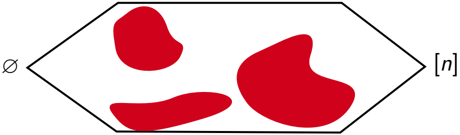
- For a partition $f : [n] \to \{\pm 1\}$, we say that $\mathcal{X}$ is k-**chain-balanced w.r.t.** f if there is a chain

$$X_0 \subsetneq X_1 \subsetneq \ldots \subsetneq X_n$$

s.t. $|X_i| = |X_{i-1}| + 1$ and $X_0 = \varnothing$ and

$$|f(X_i)| := |\sum_{x \in X_i} f(x)| \leq k.$$

If this is the case, we write $\text{cbal}(\mathcal{X}, f) \leq k$.



$R := \{ S \in \mathcal{X} \mid |f(S)| > k \}$

Set systems and their chain-balance

Formal definitions: $\mathcal{X} \subseteq \mathcal{P}([n])$ set system over $[n]$.

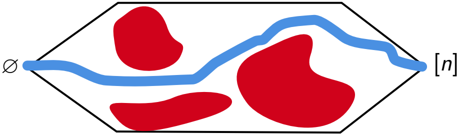
- For a partition $f : [n] \to \{\pm 1\}$, we say that $\mathcal{X}$ is k-**chain-balanced w.r.t.** f if there is a chain

$$X_0 \subsetneq X_1 \subsetneq \ldots \subsetneq X_n$$

s.t. $|X_i| = |X_{i-1}| + 1$ and $X_0 = \varnothing$ and

$$|f(X_i)| := |\sum_{x \in X_i} f(x)| \leq k.$$

If this is the case, we write $\mathrm{cbal}(\mathcal{X}, f) \leq k$.



$R := \{ S \in \mathcal{X} \mid |f(S)| > k \}$

$\mathrm{cbal}(\mathcal{X}, f) \leq k \iff \exists$blue path

Set systems and their chain-balance

Formal definitions: $\mathcal{X} \subseteq \mathcal{P}([n])$ set system over $[n]$.

- For a partition $f\colon [n] \to \{\pm 1\}$, we say that $\mathcal{X}$ is k-**chain-balanced w.r.t.** f if there is a chain

$$X_0 \subsetneq X_1 \subsetneq \ldots \subsetneq X_n$$

  s.t. $|X_i| = |X_{i-1}| + 1$ and $X_0 = \varnothing$ and

$$|f(X_i)| := |\sum_{x \in X_i} f(x)| \leq k.$$

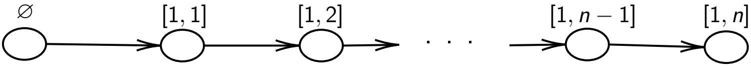
  If this is the case, we write $\mathrm{cbal}(\mathcal{X}, f) \leq k$.
- We say that $\mathcal{X}$ is k-**chain-balanced** if $\mathrm{cbal}(\mathcal{X}, f) \leq k$ for every balanced f.

Example 1: chain set system

Example 1: chain set system

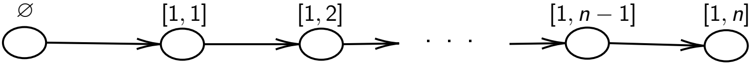
- $\mathcal{S}_n := \{\, [1, i] \mid i \in \{0, \ldots, n\}\}$ is a set system over $[n]$.

Example 1: chain set system

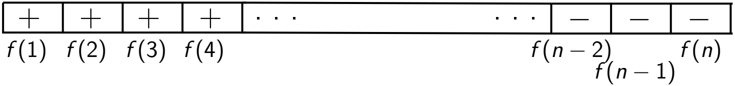


- $\mathcal{S}_n := \{\, [1,i] \mid i \in \{0,\ldots,n\}\}$ is a set system over $[n]$.

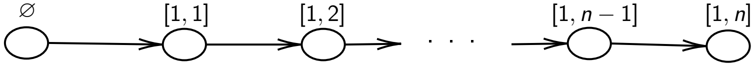
Example 1: chain set system



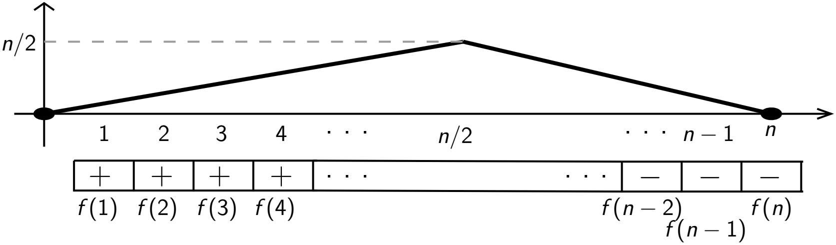
- $\mathcal{S}_n := \{\, [1,i] \mid i \in \{0,\ldots,n\}\}$ is a set system over $[n]$.



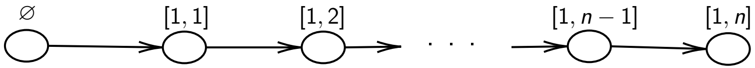
Example 1: chain set system



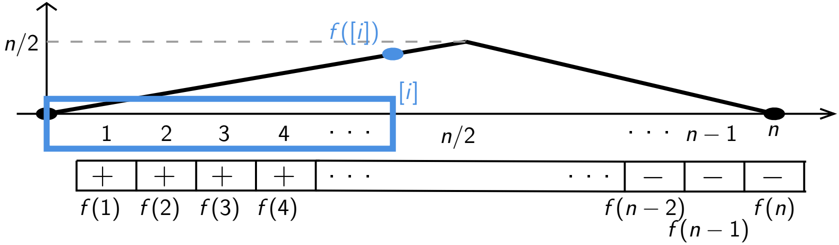
- $\mathcal{S}_n := \{\, [1,i] \mid i \in \{0,\ldots,n\}\}$ is a set system over $[n]$.



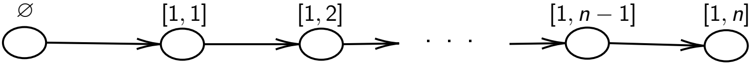
Example 1: chain set system



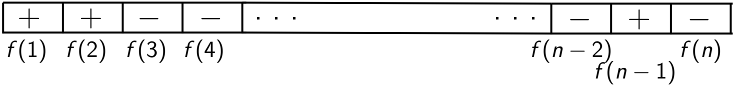
- $\mathcal{S}_n := \{\, [1,i] \mid i \in \{0,\ldots,n\}\}$ is a set system over $[n]$.



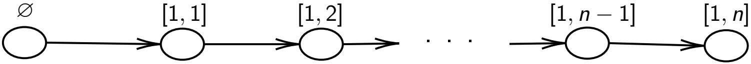
Example 1: chain set system



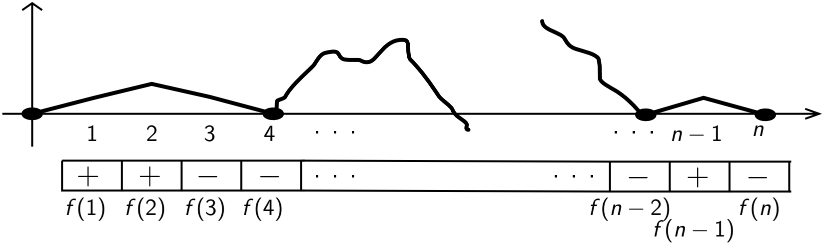
- $\mathcal{S}_n := \{\, [1,i] \mid i \in \{0,\ldots,n\}\}$ is a set system over $[n]$.



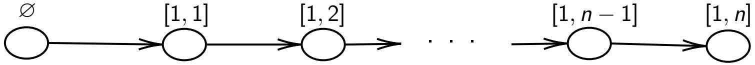
Example 1: chain set system



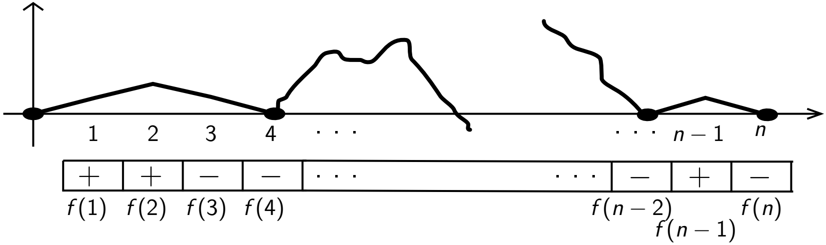
- $\mathcal{S}_n := \{\,[1,i] \mid i \in \{0,\ldots,n\}\}$ is a set system over $[n]$.



Example 1: chain set system

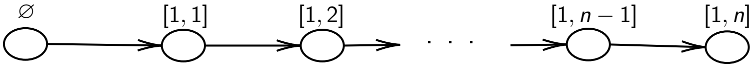


- $\mathcal{S}_n := \{\, [1,i] \mid i \in \{0,\ldots,n\}\}$ is a set system over $[n]$.



- Thus, $\Pr_f\left[\mathrm{cbal}(\mathcal{S}_n, f) \leq n^{1/2-\varepsilon}\right] \leq \exp(-n^{\Omega(1)})$.

Example 1: chain set system



- $\mathcal{S}_n := \{\,[1,i] \mid i \in \{0,\ldots,n\}\}$ is a set system over $[n]$.

- Thus, $\Pr_f\left[\mathrm{cbal}(\mathcal{S}_n, f) \leq n^{1/2-\varepsilon}\right] \leq \exp(-n^{\Omega(1)})$.
- Chain s.s. is related to CKSS'24's lower bound for Σ_π**mlABPs**: union bound over all permutations π of a given Σ_π**mlABP**.

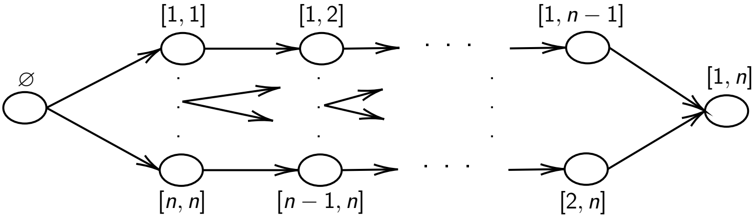
Example 2: interval set system

Example 2: interval set system

- $\mathcal{I}_n := \{\, [i,j] \mid i,j \in [n] \}$.

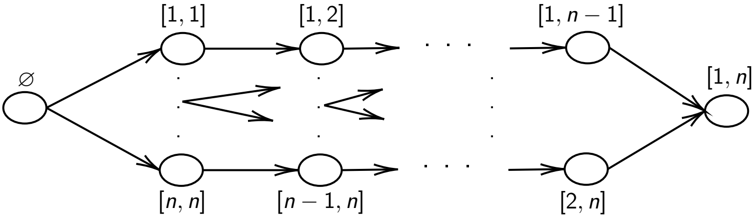
Example 2: interval set system

- $\mathcal{I}_n := \{ [i,j] \mid i,j \in [n] \}$.



Example 2: interval set system

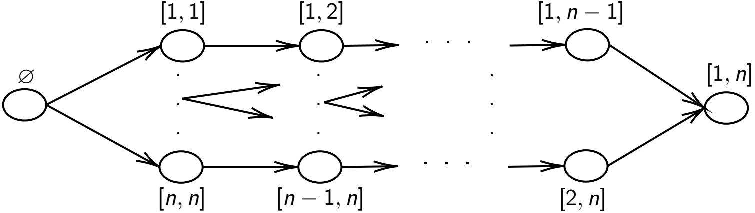
- $\mathcal{I}_n := \{ [i,j] \mid i,j \in [n] \}$.



- This is the set system corresponding to the weak-full rank mlABP constructed by DMPY'12.

Example 2: interval set system

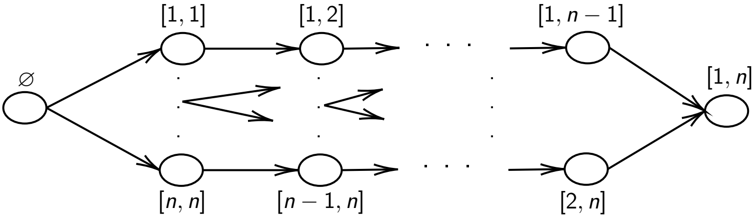
- $\mathcal{I}_n := \{\, [i,j] \mid i,j \in [n]\}$.



- This is the set system corresponding to the weak-full rank mlABP constructed by DMPY'12.
- We show $\Pr_f\left[\mathrm{cbal}(\mathcal{I}_n, f) \leq n^{1/5}\right] \leq \exp(-\Omega(n^{1/5}))$.

Example 2: interval set system

- $\mathcal{I}_n := \{\, [i,j] \mid i,j \in [n] \}$.



- This is the set system corresponding to the weak-full rank mlABP constructed by DMPY'12.
- We show $\Pr_f\left[\mathrm{cbal}(\mathcal{I}_n, f) \le n^{1/5}\right] \le \exp(-\Omega(n^{1/5}))$.
- By our framework, this implies that DMPY'12's mlABP is far from full-rank for most balanced partition.

Chain-balance lower bounds for interval set systems

Chain-balance lower bounds for interval set systems

Theorem 2

For a random balanced partition $f\colon [n] \to \{\pm 1\}$, we have

$$\Pr_f \left[\mathsf{cbal}(\mathcal{I}_n, f) \le n^{1/5}\right] \le \exp(-\Omega(n^{1/5})).$$

Chain-balance lower bounds for interval set systems

Theorem 2

For a random partition $f\colon [n] \to \{\pm 1\}$*, we have*

$$\Pr_f\left[\mathrm{cbal}(\mathcal{I}_n, f) \leq 1\right] \leq \exp(-\Omega(n^{1/5})).$$

Proof.

Chain-balance lower bounds for interval set systems

Theorem 2

For a random partition $f\colon [n] \to \{\pm 1\}$, we have

$$\Pr_f\left[\mathrm{cbal}(\mathcal{I}_n, f) \leq 1\right] \leq \exp(-\Omega(n^{1/5})).$$

Proof.

1. f is an **arbitrary** partition and suppose $\mathrm{cbal}(\mathcal{I}_n, f) \leq d := 1$.

Chain-balance lower bounds for interval set systems

Theorem 2

For a random partition $f : [n] \to \{\pm 1\}$, we have

$$\Pr_f [\mathrm{cbal}(\mathcal{I}_n, f) \leq 1] \leq \exp(-\Omega(n^{1/5})).$$

Proof.

1. f is an **arbitrary** partition and suppose $\mathrm{cbal}(\mathcal{I}_n, f) \leq d := 1$.
2. Let $I_0 \subsetneq \ldots \subsetneq I_n$ with $|f(I_i)| \leq d$ and $|I_i| = |I_{i-1}| + 1$ and $I_0 = \varnothing$.

Chain-balance lower bounds for interval set systems

Theorem 2

For a random partition $f\colon [n] \to \{\pm 1\}$, we have

$$\Pr_f [\mathrm{cbal}(\mathcal{I}_n, f) \leq 1] \leq \exp(-\Omega(n^{1/5})).$$

Proof.

1. f is an **arbitrary** partition and suppose $\mathrm{cbal}(\mathcal{I}_n, f) \leq d := 1$.
2. Let $I_0 \subsetneq \ldots \subsetneq I_n$ with $|f(I_i)| \leq d$ and $|I_i| = |I_{i-1}| + 1$ and $I_0 = \varnothing$.
3. Let $s, e \in [n]$ such that $I_1 = \{s\}$ and $I_{n-1} = [n] \setminus \{e\}$.

Chain-balance lower bounds for interval set systems

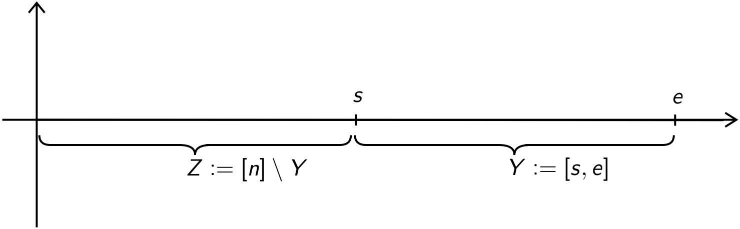
Theorem 2

For a random partition $f\colon [n] \to \{\pm 1\}$, *we have*

$$\Pr_f \left[\mathrm{cbal}(\mathcal{I}_n, f) \leq 1\right] \leq \exp(-\Omega(n^{1/5})).$$

Proof.

1. f is an **arbitrary** partition and suppose $\mathrm{cbal}(\mathcal{I}_n, f) \leq d := 1$.
2. Let $I_0 \subsetneq \ldots \subsetneq I_n$ with $|f(I_i)| \leq d$ and $|I_i| = |I_{i-1}| + 1$ and $I_0 = \varnothing$.
3. Let $s, e \in [n]$ such that $I_1 = \{s\}$ and $I_{n-1} = [n] \setminus \{e\}$.



Chain-balance lower bounds for interval set systems

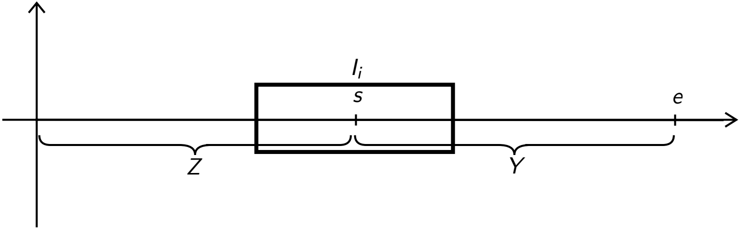
Theorem 2

For a random partition $f\colon [n] \to \{\pm 1\}$, we have

$$\Pr_f\left[\mathrm{cbal}(\mathcal{I}_n, f) \leq 1\right] \leq \exp(-\Omega(n^{1/5})).$$

Proof.

1. f is an **arbitrary** partition and suppose $\mathrm{cbal}(\mathcal{I}_n, f) \leq d := 1$.
2. Let $I_0 \subsetneq \ldots \subsetneq I_n$ with $|f(I_i)| \leq d$ and $|I_i| = |I_{i-1}| + 1$ and $I_0 = \varnothing$.
3. Let $s, e \in [n]$ such that $I_1 = \{s\}$ and $I_{n-1} = [n] \setminus \{e\}$.



Chain-balance lower bounds for interval set systems

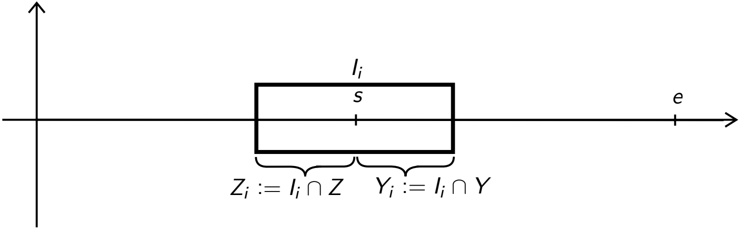
Theorem 2

For a random partition $f\colon [n] \to \{\pm 1\}$, *we have*

$$\Pr_f\left[\mathrm{cbal}(\mathcal{I}_n, f) \le 1\right] \le \exp(-\Omega(n^{1/5})).$$

Proof.

1. f is an **arbitrary** partition and suppose $\mathrm{cbal}(\mathcal{I}_n, f) \le d := 1$.
2. Let $I_0 \subsetneq \ldots \subsetneq I_n$ with $|f(I_i)| \le d$ and $|I_i| = |I_{i-1}| + 1$ and $I_0 = \varnothing$.
3. Let $s, e \in [n]$ such that $I_1 = \{s\}$ and $I_{n-1} = [n] \setminus \{e\}$.



Chain-balance lower bounds for interval set systems

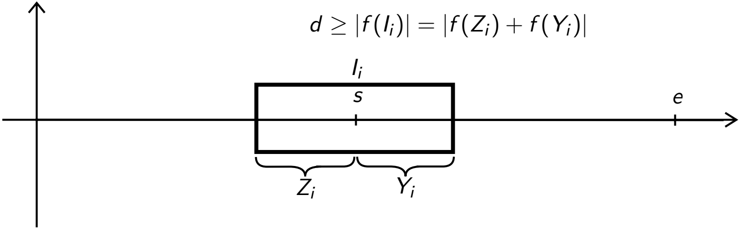
Theorem 2

For a random partition $f : [n] \to \{\pm 1\}$, *we have*

$$\Pr_f [\mathrm{cbal}(\mathcal{I}_n, f) \leq 1] \leq \exp(-\Omega(n^{1/5})).$$

Proof.

1. f is an **arbitrary** partition and suppose $\mathrm{cbal}(\mathcal{I}_n, f) \leq d := 1$.
2. Let $I_0 \subsetneq \ldots \subsetneq I_n$ with $|f(I_i)| \leq d$ and $|I_i| = |I_{i-1}| + 1$ and $I_0 = \varnothing$.
3. Let $s, e \in [n]$ such that $I_1 = \{s\}$ and $I_{n-1} = [n] \setminus \{e\}$.



Chain-balance lower bounds for interval set systems

Theorem 2

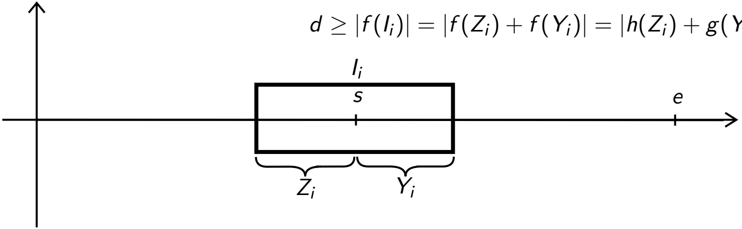
For a random partition $f : [n] \to \{\pm 1\}$, we have

$$\Pr_f \left[\mathrm{cbal}(\mathcal{I}_n, f) \le 1\right] \le \exp(-\Omega(n^{1/5})).$$

Proof.

1. f is an **arbitrary** partition and suppose $\mathrm{cbal}(\mathcal{I}_n, f) \le d := 1$.
2. Let $I_0 \subsetneq \ldots \subsetneq I_n$ with $|f(I_i)| \le d$ and $|I_i| = |I_{i-1}| + 1$ and $I_0 = \varnothing$.
3. Let $s, e \in [n]$ such that $I_1 = \{s\}$ and $I_{n-1} = [n] \setminus \{e\}$.
4. $h := f\restriction_Z$ and $g := f\restriction_Y$.

$$d \ge |f(I_i)| = |f(Z_i) + f(Y_i)| = |h(Z_i) + g(Y_i)|$$



Chain-balance lower bounds for interval set systems

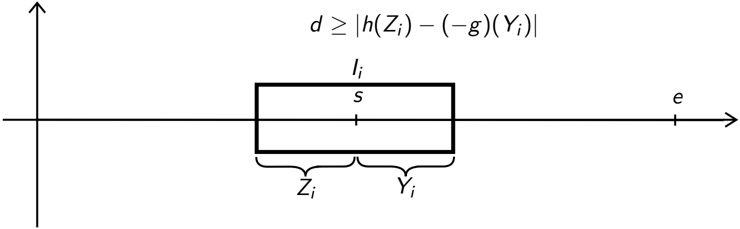
Theorem 2

For a random partition $f\colon [n] \to \{\pm 1\}$, we have

$$\Pr_f\left[\mathrm{cbal}(\mathcal{I}_n, f) \le 1\right] \le \exp(-\Omega(n^{1/5})).$$

Proof.

1. f is an **arbitrary** partition and suppose $\mathrm{cbal}(\mathcal{I}_n, f) \le d := 1$.
2. Let $I_0 \subsetneq \ldots \subsetneq I_n$ with $|f(I_i)| \le d$ and $|I_i| = |I_{i-1}| + 1$ and $I_0 = \varnothing$.
3. Let $s, e \in [n]$ such that $I_1 = \{s\}$ and $I_{n-1} = [n] \setminus \{e\}$.
4. $h := f\restriction_Z$ and $g := f\restriction_Y$.



Chain-balance lower bounds for interval set systems

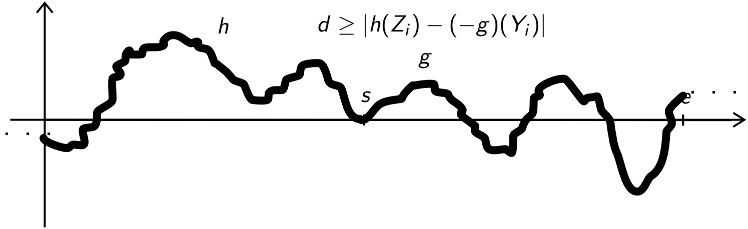
Theorem 2

For a random partition $f\colon [n] \to \{\pm 1\}$, we have

$$\Pr_f\left[\mathrm{cbal}(\mathcal{I}_n, f) \le 1\right] \le \exp(-\Omega(n^{1/5})).$$

Proof.

1. f is an **arbitrary** partition and suppose $\mathrm{cbal}(\mathcal{I}_n, f) \le d := 1$.
2. Let $I_0 \subsetneq \ldots \subsetneq I_n$ with $|f(I_i)| \le d$ and $|I_i| = |I_{i-1}| + 1$ and $I_0 = \varnothing$.
3. Let $s, e \in [n]$ such that $I_1 = \{s\}$ and $I_{n-1} = [n] \setminus \{e\}$.
4. $h := f\restriction_Z$ and $g := f\restriction_Y$.



Chain-balance lower bounds for interval set systems

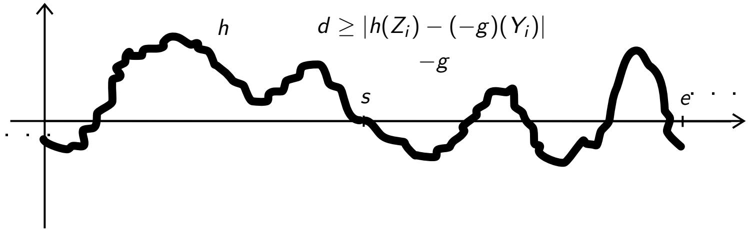
Theorem 2

For a random partition $f\colon [n] \to \{\pm 1\}$, *we have*

$$\Pr_f\left[\mathrm{cbal}(\mathcal{I}_n, f) \le 1\right] \le \exp(-\Omega(n^{1/5})).$$

Proof.

1. f is an **arbitrary** partition and suppose $\mathrm{cbal}(\mathcal{I}_n, f) \le d := 1$.
2. Let $I_0 \subsetneq \ldots \subsetneq I_n$ with $|f(I_i)| \le d$ and $|I_i| = |I_{i-1}| + 1$ and $I_0 = \varnothing$.
3. Let $s, e \in [n]$ such that $I_1 = \{s\}$ and $I_{n-1} = [n] \setminus \{e\}$.
4. $h := f\restriction_Z$ and $g := f\restriction_Y$.



Chain-balance lower bounds for interval set systems

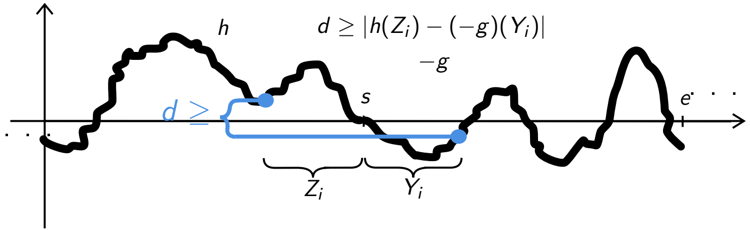
Theorem 2

For a random partition $f\colon [n] \to \{\pm 1\}$, *we have*

$$\Pr_f\left[\mathrm{cbal}(\mathcal{I}_n, f) \le 1\right] \le \exp(-\Omega(n^{1/5})).$$

Proof.

1. f is an **arbitrary** partition and suppose $\mathrm{cbal}(\mathcal{I}_n, f) \le d := 1$.
2. Let $I_0 \subsetneq \ldots \subsetneq I_n$ with $|f(I_i)| \le d$ and $|I_i| = |I_{i-1}| + 1$ and $I_0 = \varnothing$.
3. Let $s, e \in [n]$ such that $I_1 = \{s\}$ and $I_{n-1} = [n] \setminus \{e\}$.
4. $h := f\restriction_Z$ and $g := f\restriction_Y$.



Chain-balance lower bounds for interval set systems

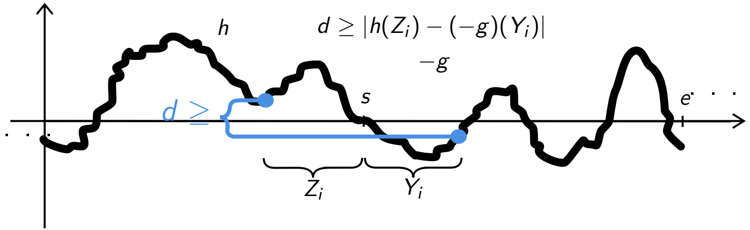
Theorem 2

For a random partition $f : [n] \to \{\pm 1\}$, we have

$$\Pr_f \left[\mathrm{cbal}(\mathcal{I}_n, f) \leq 1\right] \leq \exp(-\Omega(n^{1/5})).$$

Proof.

1. f is an **arbitrary** partition and suppose $\mathrm{cbal}(\mathcal{I}_n, f) \leq d := 1$.
2. Let $I_0 \subsetneq \ldots \subsetneq I_n$ with $|f(I_i)| \leq d$ and $|I_i| = |I_{i-1}| + 1$ and $I_0 = \varnothing$.
3. Let $s, e \in [n]$ such that $I_1 = \{s\}$ and $I_{n-1} = [n] \setminus \{e\}$.
4. $h := f{\restriction}_Z$ and $g := f{\restriction}_Y$. Then (Fréchet distance btw. h and $-g$) $\leq d$.



Nontrivial construction of 1-chain-balanced set systems

Nontrivial construction of 1-chain-balanced set systems

Theorem 3

There is a 1-chain-balanced set system $\mathcal{X}$ for $[n]$ of size $n^{O(\ln n/\ln\ln n)}$.

Nontrivial construction of 1-chain-balanced set systems

Theorem 3

There is a 1-chain-balanced set system $\mathcal{X}$ for $[n]$ of size $n^{O(\ln n / \ln \ln n)}$.

Goal: Find $\mathcal{X}$ s.t. $\forall$ balanced $f\colon [n] \to \{\pm 1\}$ $\exists X_0 \subsetneq \ldots \subsetneq X_n$ in $\mathcal{X}$ with

$$|f(X_i)| \leq 1 \text{ and } |X_i| = |X_{i-1}| + 1 \text{ and } X_0 = \varnothing.$$

Nontrivial construction of 1-chain-balanced set systems

Goal: $S(n) := \min\{\, |\mathcal{A}| \mid \mathcal{A} \text{ is a 1-cbal. s.s. for } [n]\} \leq n^{O(\ln n / \ln \ln n)}$.

Nontrivial construction of 1-chain-balanced set systems

Goal: $S(n) := \min\{\, |\mathcal{A}| \mid \mathcal{A} \text{ is a 1-cbal. s.s. for } [n]\} \leq n^{O(\ln n/\ln\ln n)}$.

Excursion Lemma (Csáki, Erdős, Révész'85): For a $\frac{1}{n}$-fraction of balanced f, we can use f to find $J_0 \subsetneq \ldots \subsetneq J_\ell$ in the chain s.s. $\mathcal{S}_n$ s.t.

$$|f(J_i)| = 0 \text{ and } J_0 = \varnothing \text{ and } |J_i| - |J_{i-1}| \leq \lambda := \frac{n}{\ln n}$$

Nontrivial construction of 1-chain-balanced set systems

Goal: $S(n) := \min\{\, |\mathcal{A}| \mid \mathcal{A} \text{ is a 1-cbal. s.s. for } [n]\} \leq n^{O(\ln n / \ln\ln n)}$.

Excursion Lemma (Csáki, Erdős, Révész'85): For a $\frac{1}{n}$-fraction of balanced f, we can use f to find $J_0 \subsetneq \ldots \subsetneq J_\ell$ in the chain s.s. $\mathcal{S}_n$ s.t.

$$|f(J_i)| = 0 \text{ and } J_0 = \varnothing \text{ and } |J_i| - |J_{i-1}| \leq \lambda := \frac{n}{\ln n}$$

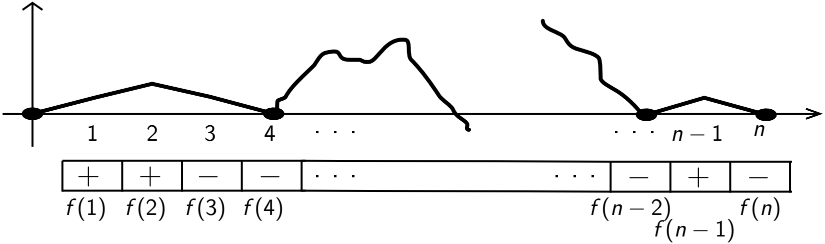
$+$	$+$	$-$	$-$	$\cdots$	$\cdots$	$-$	$+$	$-$
$f(1)$	$f(2)$	$f(3)$	$f(4)$			$f(n-2)$	$f(n-1)$	$f(n)$

Nontrivial construction of 1-chain-balanced set systems

Goal: $S(n) := \min\{\, |\mathcal{A}| \mid \mathcal{A} \text{ is a 1-cbal. s.s. for } [n]\} \leq n^{O(\ln n/\ln\ln n)}$.

Excursion Lemma (Csáki, Erdős, Révész'85): For a $\frac{1}{n}$-fraction of balanced f, we can use f to find $J_0 \subsetneq \ldots \subsetneq J_\ell$ in the chain s.s. $\mathcal{S}_n$ s.t.

$$|f(J_i)| = 0 \text{ and } J_0 = \varnothing \text{ and } |J_i| - |J_{i-1}| \leq \lambda := \frac{n}{\ln n}$$

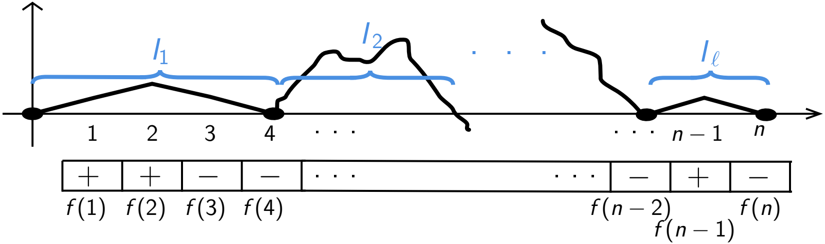


Nontrivial construction of 1-chain-balanced set systems

Goal: $S(n) := \min\{\, |\mathcal{A}| \mid \mathcal{A} \text{ is a 1-cbal. s.s. for } [n]\} \leq n^{O(\ln n/\ln\ln n)}$.

Excursion Lemma (Csáki, Erdős, Révész'85): For a $\frac{1}{n}$-fraction of balanced f, we can use f to find $J_0 \subsetneq \ldots \subsetneq J_\ell$ in the chain s.s. $\mathcal{S}_n$ s.t.

$$|f(J_i)| = 0 \text{ and } J_0 = \varnothing \text{ and } |J_i| - |J_{i-1}| \leq \lambda := \frac{n}{\ln n}$$

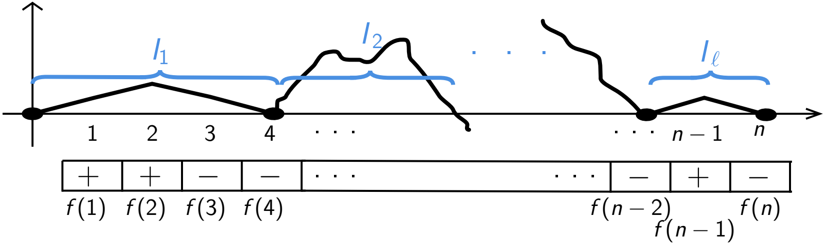


Nontrivial construction of 1-chain-balanced set systems

Goal: $S(n) := \min\{\, |\mathcal{A}| \mid \mathcal{A} \text{ is a 1-cbal. s.s. for } [n]\} \leq n^{O(\ln n/\ln\ln n)}$.

Excursion Lemma (Csáki, Erdős, Révész'85): For a $\frac{1}{n}$-fraction of balanced f, we can use f to find $J_0 \subsetneq \ldots \subsetneq J_\ell$ in the chain s.s. $\mathcal{S}_n$ s.t.

$$|f(J_i)| = 0 \text{ and } J_0 = \varnothing \text{ and } |J_i| - |J_{i-1}| \leq \lambda := \frac{n}{\ln n} \text{ and } J_i := \cup_{j\leq i} I_j.$$



Nontrivial construction of 1-chain-balanced set systems

Goal: $S(n) := \min\{\, |\mathcal{A}| \mid \mathcal{A}$ is a 1-cbal. s.s. for $[n]\} \le n^{O(\ln n / \ln \ln n)}$.

Excursion Lemma (Csáki, Erdős, Révész'85): For a $\frac{1}{n}$-fraction of balanced f, we can use f to find $J_0 \subsetneq \ldots \subsetneq J_\ell$ in the chain s.s. $\mathcal{S}_n$ s.t.

$$|f(J_i)| = 0 \text{ and } J_0 = \varnothing \text{ and } |J_i| - |J_{i-1}| \le \lambda := \frac{n}{\ln n} \text{ and } J_i := \cup_{j \le i} I_j.$$

Chain Covering Lemma: There are $L := n^2$ permutations $\pi_1, \cdots, \pi_L$ s.t., for all balanced f, $\exists k \in [L], \exists J_0 \subsetneq \ldots \subsetneq J_\ell$ in $\pi_k \circ \mathcal{S}_n$ s.t.

$$|f(J_i)| = 0 \text{ and } |J_i| \le |J_{i-1}| + \lambda \text{ and } J_0 = \varnothing.$$

Nontrivial construction of 1-chain-balanced set systems

Goal: $S(n) := \min\{ |\mathcal{A}| \mid \mathcal{A} \text{ is a 1-cbal. s.s. for } [n]\} \leq n^{O(\ln n / \ln \ln n)}$.

Excursion Lemma (Csáki, Erdős, Révész'85): For a $\frac{1}{n}$-fraction of balanced f, we can use f to find $J_0 \subsetneq \ldots \subsetneq J_\ell$ in the chain s.s. $\mathcal{S}_n$ s.t.

$$|f(J_i)| = 0 \text{ and } J_0 = \varnothing \text{ and } |J_i| - |J_{i-1}| \leq \lambda := \frac{n}{\ln n} \text{ and } J_i := \cup_{j \leq i} I_j.$$

Chain Covering Lemma: There are $L := n^2$ permutations $\pi_1, \cdots, \pi_L$ s.t., for all balanced f, $\exists k \in [L], \exists J_0 \subsetneq \ldots \subsetneq J_\ell$ in $\pi_k \circ \mathcal{S}_n$ s.t.

$$|f(J_i)| = 0 \text{ and } |J_i| \leq |J_{i-1}| + \lambda \text{ and } J_0 = \varnothing.$$

Set system recursive construction:

Nontrivial construction of 1-chain-balanced set systems

Goal: $S(n) := \min\{ |\mathcal{A}| \mid \mathcal{A} \text{ is a 1-cbal. s.s. for } [n]\} \leq n^{O(\ln n/\ln\ln n)}$.

Excursion Lemma (Csáki, Erdős, Révész'85): For a $\frac{1}{n}$-fraction of balanced f, we can use f to find $J_0 \subsetneq \ldots \subsetneq J_\ell$ in the chain s.s. $\mathcal{S}_n$ s.t.

$$|f(J_i)| = 0 \text{ and } J_0 = \varnothing \text{ and } |J_i| - |J_{i-1}| \leq \lambda := \frac{n}{\ln n} \text{ and } J_i := \cup_{j \leq i} I_j.$$

Chain Covering Lemma: There are $L := n^2$ permutations $\pi_1, \cdots, \pi_L$ s.t., for all balanced f, $\exists k \in [L], \exists J_0 \subsetneq \ldots \subsetneq J_\ell$ in $\pi_k \circ \mathcal{S}_n$ s.t.

$$|f(J_i)| = 0 \text{ and } |J_i| \leq |J_{i-1}| + \lambda \text{ and } J_0 = \varnothing.$$

Set system recursive construction:

1. Apply Chain Covering Lemma to obtain n^2 chain set systems $\pi_i \circ \mathcal{S}_n$.

Nontrivial construction of 1-chain-balanced set systems

Goal: $S(n) := \min\{\, |\mathcal{A}| \mid \mathcal{A} \text{ is a 1-cbal. s.s. for } [n]\} \leq n^{O(\ln n/\ln\ln n)}$.

Excursion Lemma (Csáki, Erdős, Révész'85): For a $\frac{1}{n}$-fraction of balanced f, we can use f to find $J_0 \subsetneq \ldots \subsetneq J_\ell$ in the chain s.s. $\mathcal{S}_n$ s.t.

$$|f(J_i)| = 0 \text{ and } J_0 = \varnothing \text{ and } |J_i| - |J_{i-1}| \leq \lambda := \frac{n}{\ln n} \text{ and } J_i := \cup_{j \leq i} I_j.$$

Chain Covering Lemma: There are $L := n^2$ permutations $\pi_1, \cdots, \pi_L$ s.t., for all balanced f, $\exists k \in [L], \exists J_0 \subsetneq \ldots \subsetneq J_\ell$ in $\pi_k \circ \mathcal{S}_n$ s.t.

$$|f(J_i)| = 0 \text{ and } |J_i| \leq |J_{i-1}| + \lambda \text{ and } J_0 = \varnothing.$$

Set system recursive construction:

1. Apply Chain Covering Lemma to obtain n^2 chain set systems $\pi_i \circ \mathcal{S}_n$.
2. Fill the gaps of length at most λ using an optimal s.s. for $S(\lambda)$.

Nontrivial construction of 1-chain-balanced set systems

Goal: $S(n) := \min\{\, |\mathcal{A}| \mid \mathcal{A} \text{ is a 1-cbal. s.s. for } [n]\} \leq n^{O(\ln n/\ln\ln n)}$.

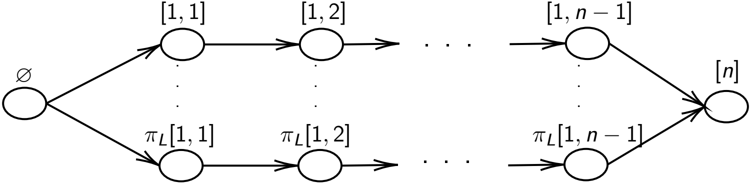
Excursion Lemma (Csáki, Erdős, Révész'85): For a $\frac{1}{n}$-fraction of balanced f, we can use f to find $J_0 \subsetneq \ldots \subsetneq J_\ell$ in the chain s.s. $\mathcal{S}_n$ s.t.

$$|f(J_i)| = 0 \text{ and } J_0 = \varnothing \text{ and } |J_i| - |J_{i-1}| \leq \lambda := \frac{n}{\ln n} \text{ and } J_i := \cup_{j \leq i} I_j.$$

Chain Covering Lemma: There are $L := n^2$ permutations $\pi_1, \cdots, \pi_L$ s.t., for all balanced f, $\exists k \in [L], \exists J_0 \subsetneq \ldots \subsetneq J_\ell$ in $\pi_k \circ \mathcal{S}_n$ s.t.

$$|f(J_i)| = 0 \text{ and } |J_i| \leq |J_{i-1}| + \lambda \text{ and } J_0 = \varnothing.$$

Set system recursive construction:



Nontrivial construction of 1-chain-balanced set systems

Goal: $S(n) := \min\{\, |\mathcal{A}| \mid \mathcal{A} \text{ is a 1-cbal. s.s. for } [n]\} \leq n^{O(\ln n / \ln \ln n)}$.

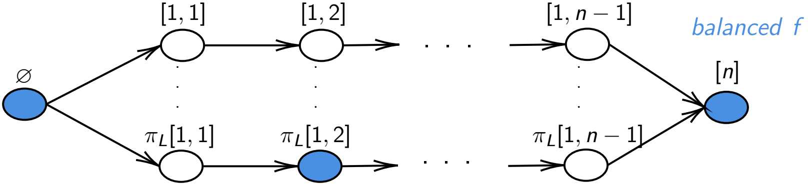
Excursion Lemma (Csáki, Erdős, Révész'85): For a $\frac{1}{n}$-fraction of balanced f, we can use f to find $J_0 \subsetneq \ldots \subsetneq J_\ell$ in the chain s.s. $\mathcal{S}_n$ s.t.

$$|f(J_i)| = 0 \text{ and } J_0 = \varnothing \text{ and } |J_i| - |J_{i-1}| \leq \lambda := \frac{n}{\ln n} \text{ and } J_i := \cup_{j \leq i} I_j.$$

Chain Covering Lemma: There are $L := n^2$ permutations $\pi_1, \cdots, \pi_L$ s.t., for all balanced f, $\exists k \in [L], \exists J_0 \subsetneq \ldots \subsetneq J_\ell$ in $\pi_k \circ \mathcal{S}_n$ s.t.

$$|f(J_i)| = 0 \text{ and } |J_i| \leq |J_{i-1}| + \lambda \text{ and } J_0 = \varnothing.$$

Set system recursive construction:



Nontrivial construction of 1-chain-balanced set systems

Goal: $S(n) := \min\{ |\mathcal{A}| \mid \mathcal{A} \text{ is a 1-cbal. s.s. for } [n]\} \leq n^{O(\ln n/\ln\ln n)}$.

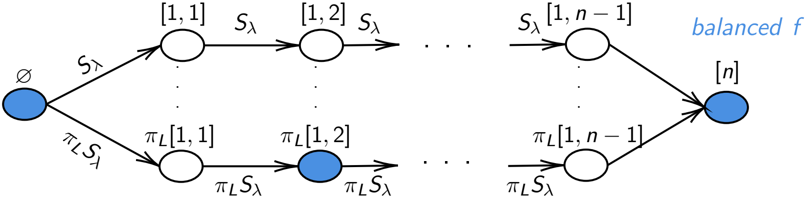
Excursion Lemma (Csáki, Erdős, Révész'85): For a $\frac{1}{n}$-fraction of balanced f, we can use f to find $J_0 \subsetneq \ldots \subsetneq J_\ell$ in the chain s.s. $\mathcal{S}_n$ s.t.

$$|f(J_i)| = 0 \text{ and } J_0 = \varnothing \text{ and } |J_i| - |J_{i-1}| \leq \lambda := \frac{n}{\ln n} \text{ and } J_i := \cup_{j\leq i} I_j.$$

Chain Covering Lemma: There are $L := n^2$ permutations $\pi_1, \cdots, \pi_L$ s.t., for all balanced f, $\exists k \in [L], \exists J_0 \subsetneq \ldots \subsetneq J_\ell$ in $\pi_k \circ \mathcal{S}_n$ s.t.

$$|f(J_i)| = 0 \text{ and } |J_i| \leq |J_{i-1}| + \lambda \text{ and } J_0 = \varnothing.$$

Set system recursive construction:



Nontrivial construction of 1-chain-balanced set systems

Goal: $S(n) := \min\{\,|\mathcal{A}| \mid \mathcal{A}$ is a 1-cbal. s.s. for $[n]\} \leq n^{O(\ln n/\ln\ln n)}$.

Excursion Lemma (Csáki, Erdős, Révész'85): For a $\frac{1}{n}$-fraction of balanced f, we can use f to find $J_0 \subsetneq \ldots \subsetneq J_\ell$ in the chain s.s. $\mathcal{S}_n$ s.t.

$$|f(J_i)| = 0 \text{ and } J_0 = \varnothing \text{ and } |J_i| - |J_{i-1}| \leq \lambda := \frac{n}{\ln n} \text{ and } J_i := \cup_{j\leq i} I_j.$$

Chain Covering Lemma: There are $L := n^2$ permutations $\pi_1, \cdots, \pi_L$ s.t., for all balanced f, $\exists k \in [L], \exists J_0 \subsetneq \ldots \subsetneq J_\ell$ in $\pi_k \circ \mathcal{S}_n$ s.t.

$$|f(J_i)| = 0 \text{ and } |J_i| \leq |J_{i-1}| + \lambda \text{ and } J_0 = \varnothing.$$

Set system recursive construction:

1. Apply Chain Covering Lemma to obtain n^2 chain set systems $\pi_i \circ \mathcal{S}_n$.
2. Fill the gaps of length at most λ using an optimal s.s. for $S(\lambda)$.
3. Number of sets added in total is $(n^2 \cdot n) \cdot (n \cdot S(\lambda))$.

Nontrivial construction of 1-chain-balanced set systems

Goal: $S(n) := \min\{\, |\mathcal{A}| \mid \mathcal{A} \text{ is a 1-cbal. s.s. for } [n]\} \leq n^{O(\ln n/\ln\ln n)}$.

Excursion Lemma (Csáki, Erdős, Révész'85): For a $\frac{1}{n}$-fraction of balanced f, we can use f to find $J_0 \subsetneq \ldots \subsetneq J_\ell$ in the chain s.s. $\mathcal{S}_n$ s.t.

$$|f(J_i)| = 0 \text{ and } J_0 = \varnothing \text{ and } |J_i| - |J_{i-1}| \leq \lambda := \frac{n}{\ln n} \text{ and } J_i := \cup_{j \leq i} I_j.$$

Chain Covering Lemma: There are $L := n^2$ permutations $\pi_1, \cdots, \pi_L$ s.t., for all balanced f, $\exists k \in [L], \exists J_0 \subsetneq \ldots \subsetneq J_\ell$ in $\pi_k \circ \mathcal{S}_n$ s.t.

$$|f(J_i)| = 0 \text{ and } |J_i| \leq |J_{i-1}| + \lambda \text{ and } J_0 = \varnothing.$$

Set system recursive construction:

1. Apply Chain Covering Lemma to obtain n^2 chain set systems $\pi_i \circ \mathcal{S}_n$.
2. Fill the gaps of length at most λ using an optimal s.s. for $S(\lambda)$.
3. Number of sets added in total is $(n^2 \cdot n) \cdot (n \cdot S(\lambda))$.
4. Thus, we get a 1-chain-balanced s.s. $\mathcal{X}$ s.t. $|\mathcal{X}| \leq n^d S(n/\ln n)$.

Nontrivial construction of 1-chain-balanced set systems

Goal: $S(n) := \min\{\, |\mathcal{A}| \mid \mathcal{A}$ is a 1-cbal. s.s. for $[n]\} \leq n^{O(\ln n/\ln\ln n)}$.

Excursion Lemma (Csáki, Erdős, Révész'85): For a $\frac{1}{n}$-fraction of balanced f, we can use f to find $J_0 \subsetneq \ldots \subsetneq J_\ell$ in the chain s.s. $\mathcal{S}_n$ s.t.

$$|f(J_i)| = 0 \text{ and } J_0 = \varnothing \text{ and } |J_i| - |J_{i-1}| \leq \lambda := \frac{n}{\ln n} \text{ and } J_i := \cup_{j\leq i} I_j.$$

Chain Covering Lemma: There are $L := n^2$ permutations $\pi_1, \cdots, \pi_L$ s.t., for all balanced f, $\exists k \in [L], \exists J_0 \subsetneq \ldots \subsetneq J_\ell$ in $\pi_k \circ \mathcal{S}_n$ s.t.

$$|f(J_i)| = 0 \text{ and } |J_i| \leq |J_{i-1}| + \lambda \text{ and } J_0 = \varnothing.$$

Set system recursive construction:

1. Apply Chain Covering Lemma to obtain n^2 chain set systems $\pi_i \circ \mathcal{S}_n$.
2. Fill the gaps of length at most λ using an optimal s.s. for $S(\lambda)$.
3. Number of sets added in total is $(n^2 \cdot n) \cdot (n \cdot S(\lambda))$.
4. Thus, we get a 1-chain-balanced s.s. $\mathcal{X}$ s.t. $|\mathcal{X}| \leq n^d S(n/\ln n)$.
5. Therefore, $S(n) \leq n^d S(n/\ln n)$

Nontrivial construction of 1-chain-balanced set systems

Goal: $S(n) := \min\{\, |\mathcal{A}| \mid \mathcal{A} \text{ is a 1-cbal. s.s. for } [n]\} \leq n^{O(\ln n/\ln\ln n)}$.

Excursion Lemma (Csáki, Erdős, Révész'85): For a $\frac{1}{n}$-fraction of balanced f, we can use f to find $J_0 \subsetneq \ldots \subsetneq J_\ell$ in the chain s.s. $\mathcal{S}_n$ s.t.

$$|f(J_i)| = 0 \text{ and } J_0 = \varnothing \text{ and } |J_i| - |J_{i-1}| \leq \lambda := \frac{n}{\ln n} \text{ and } J_i := \cup_{j \leq i} I_j.$$

Chain Covering Lemma: There are $L := n^2$ permutations $\pi_1, \cdots, \pi_L$ s.t., for all balanced f, $\exists k \in [L], \exists J_0 \subsetneq \ldots \subsetneq J_\ell$ in $\pi_k \circ \mathcal{S}_n$ s.t.

$$|f(J_i)| = 0 \text{ and } |J_i| \leq |J_{i-1}| + \lambda \text{ and } J_0 = \varnothing.$$

Set system recursive construction:

1. Apply Chain Covering Lemma to obtain n^2 chain set systems $\pi_i \circ \mathcal{S}_n$.
2. Fill the gaps of length at most λ using an optimal s.s. for $S(\lambda)$.
3. Number of sets added in total is $(n^2 \cdot n) \cdot (n \cdot S(\lambda))$.
4. Thus, we get a 1-chain-balanced s.s. $\mathcal{X}$ s.t. $|\mathcal{X}| \leq n^d S(n/\ln n)$.
5. Therefore, $S(n) \leq n^d S(n/\ln n)$ and $S(n) \leq n^{O(\ln n/\ln\ln n)}$.

Conclusions and open problems

Conclusions and open problems

Summary of our results

- Combinatorial "characterization" of the smallest mlABP computing a full rank polynomial via the chain-balance of set systems.

Conclusions and open problems

Summary of our results

- Combinatorial "characterization" of the smallest mlABP computing a full rank polynomial via the chain-balance of set systems.
- Lower bounds for the chain-balance of interval set systems and exponential lower bounds for restricted mlABPs.

Conclusions and open problems

Summary of our results

- Combinatorial "characterization" of the smallest mlABP computing a full rank polynomial via the chain-balance of set systems.
- Lower bounds for the chain-balance of interval set systems and exponential lower bounds for restricted mlABPs.
- Construction of a 1-chain-balanced set system of size $n^{O(\ln n / \ln \ln n)}$ and superpolynomial separation between multilinear formulas and mlABPs.

Conclusions and open problems

Summary of our results

- Combinatorial “characterization” of the smallest mlABP computing a full rank polynomial via the chain-balance of set systems.
- Lower bounds for the chain-balance of interval set systems and exponential lower bounds for restricted mlABPs.
- Construction of a 1-chain-balanced set system of size $n^{O(\ln n/\ln\ln n)}$ and superpolynomial separation between multilinear formulas and mlABPs.

Open problems

Conclusions and open problems

Summary of our results

- Combinatorial "characterization" of the smallest mlABP computing a full rank polynomial via the chain-balance of set systems.
- Lower bounds for the chain-balance of interval set systems and exponential lower bounds for restricted mlABPs.
- Construction of a 1-chain-balanced set system of size $n^{O(\ln n/\ln\ln n)}$ and superpolynomial separation between multilinear formulas and mlABPs.

Open problems

- Close the gap between $\Omega(n^2)$ (by AKV'20) and $n^{O(\ln n/\ln\ln n)}$ for the size of 1-chain-balanced set systems.

Conclusions and open problems

Summary of our results

- Combinatorial "characterization" of the smallest mlABP computing a full rank polynomial via the chain-balance of set systems.
- Lower bounds for the chain-balance of interval set systems and exponential lower bounds for restricted mlABPs.
- Construction of a 1-chain-balanced set system of size $n^{O(\ln n / \ln \ln n)}$ and superpolynomial separation between multilinear formulas and mlABPs.

Open problems

- Close the gap between $\Omega(n^2)$ (by AKV'20) and $n^{O(\ln n / \ln \ln n)}$ for the size of 1-chain-balanced set systems.
- **Explicit construction** of a 1-chain-balanced set system of size $n^{O(\ln n / \ln \ln n)}$.

Conclusions and open problems

Summary of our results

- Combinatorial "characterization" of the smallest mlABP computing a full rank polynomial via the chain-balance of set systems.
- Lower bounds for the chain-balance of interval set systems and exponential lower bounds for restricted mlABPs.
- Construction of a 1-chain-balanced set system of size $n^{O(\ln n/\ln\ln n)}$ and superpolynomial separation between multilinear formulas and mlABPs.

Open problems

- Close the gap between $\Omega(n^2)$ (by AKV'20) and $n^{O(\ln n/\ln\ln n)}$ for the size of 1-chain-balanced set systems.
- **Explicit construction** of a 1-chain-balanced set system of size $n^{O(\ln n/\ln\ln n)}$.

Thank you! Any questions?