# Optimal Depth-Three Circuits for Inner Product


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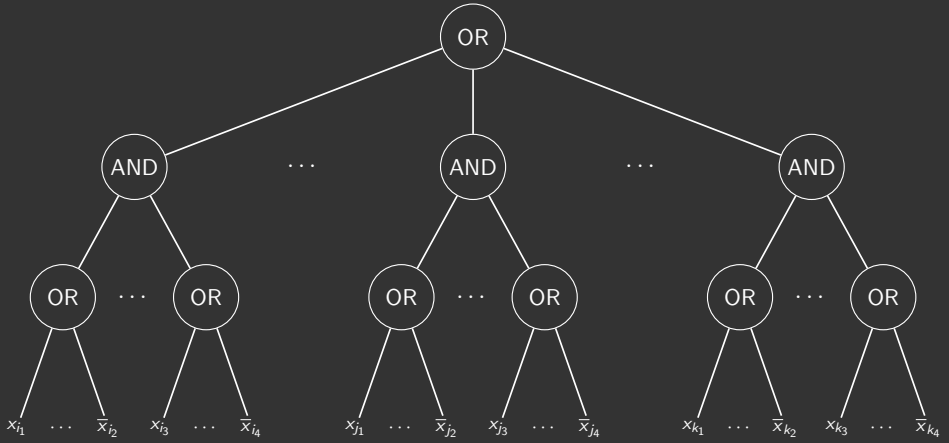
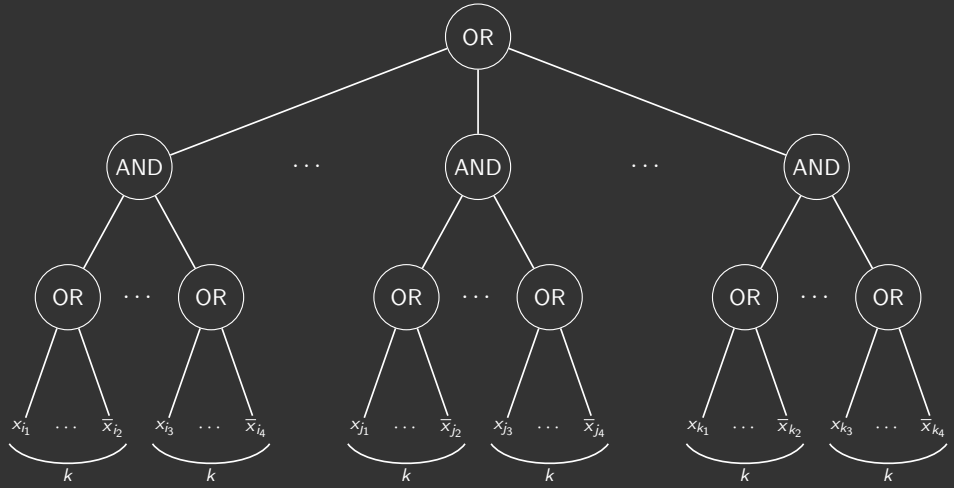
# Introduction

# Depth 3 Circuits




$\Sigma\Pi\Sigma$ circuit

# Depth 3 Circuits with fan in $k$

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**$\Sigma\Pi\Sigma_k$ complexity of $f$ (upto $2^{o(n)}$ factors)**

For $f : \{0,1\}^n \to \{0,1\}$, what is **minimum** sized $\Sigma\Pi\Sigma_k$ circuit $C$ computing $f$?

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To analyze OR of $k$-CNFs, only need to analyze $k$-CNFs

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**$k$-CNF Circuit analysis has proven useful**

Usually yields $k$-SAT algorithms e.g. Paturi-Pudlák-Saks-Zane'05

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## The best lower bound

Paturi-Púdlak-Saks-Zane'05 : $\Sigma\Pi\Sigma_k$ complexity of indicator of code is $\geq 2^{1.6 \cdot n/k}$


Inner Product

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# Problem at Hand

# $\Sigma\Pi\Sigma_2$-complexity of Inner Product

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Let $\Sigma\Pi\Sigma_2$ complexity of IP be $2^{\delta n}$

**Previous Works**

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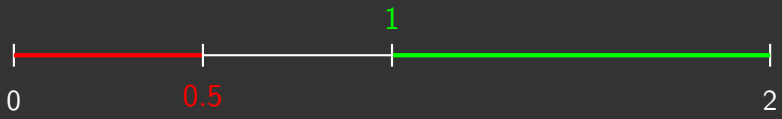

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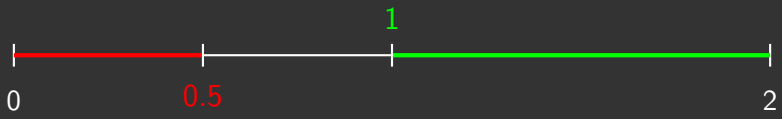

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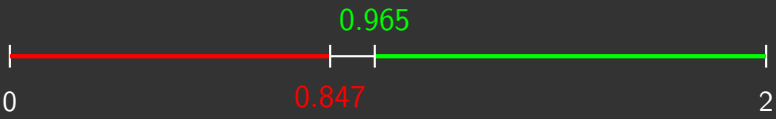

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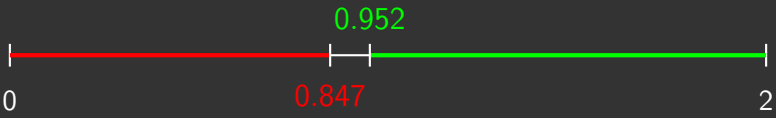

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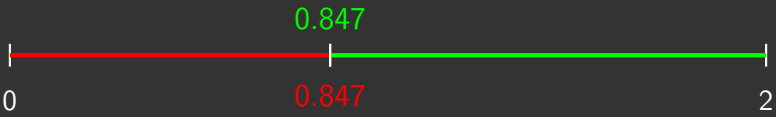
**Our Result**

$\Sigma\Pi\Sigma_2$ complexity of IP is $2^{0.847...n} = (1.8)^n$ (upto poly($n$) factors)



# Constructing Optimal Circuits for IP

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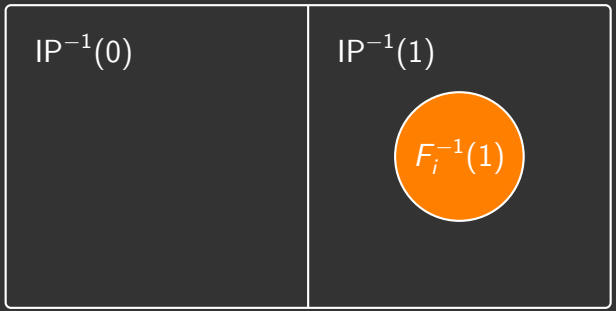

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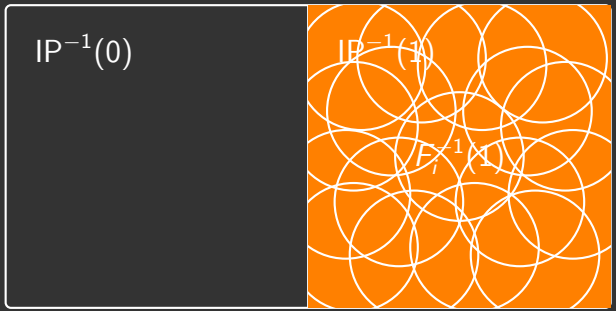

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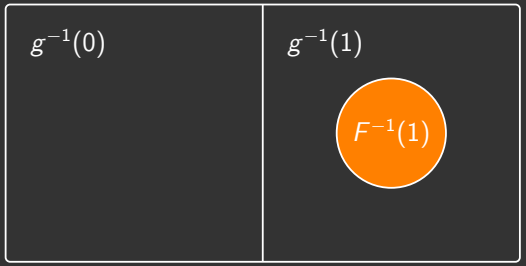

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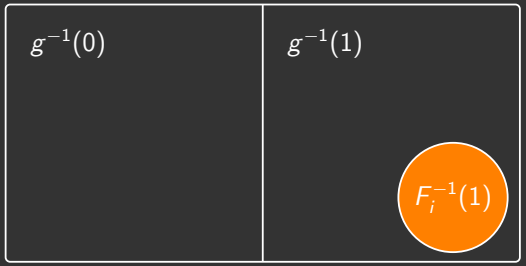

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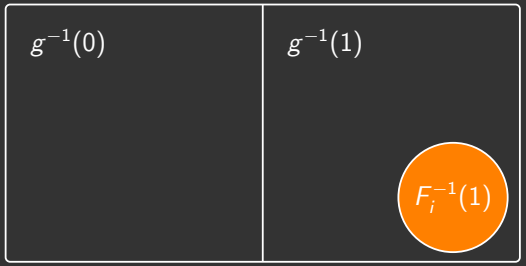

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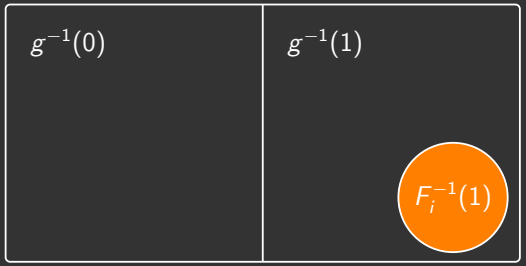

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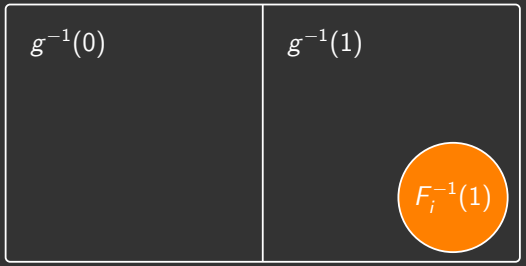
- For $\alpha \in g^{-1}(1)$, $\mathbf{Pr[\alpha \in F_i^{-1}(1)] = \frac{|F^{-1}(1)|}{|g^{-1}(1)|}}$
- Final circuit $\bigvee_{i=1}^{s} F_i$, where $\boldsymbol{s = \frac{|g^{-1}(1)|}{|F^{-1}(1)|} \cdot \mathrm{poly}(n)}$



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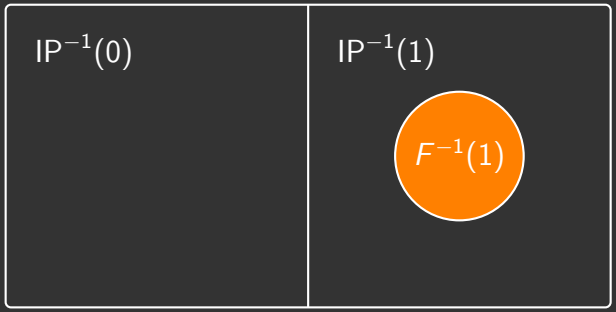

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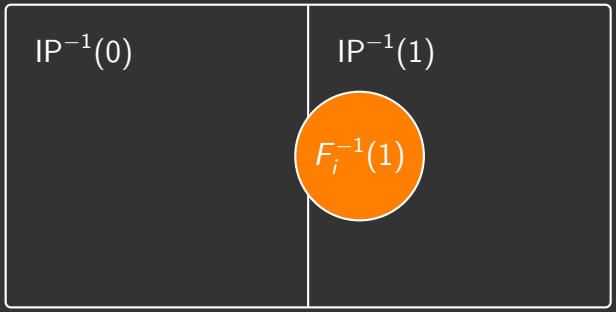

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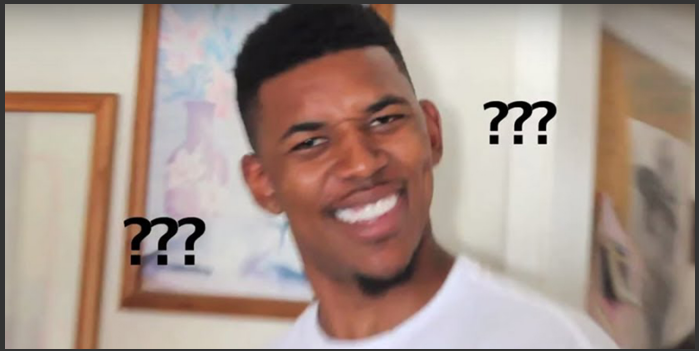
- $F_i$ MIGHT NOT be consistent with IP



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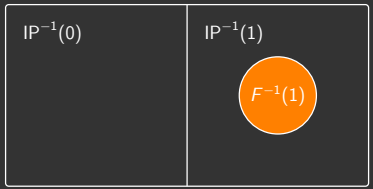

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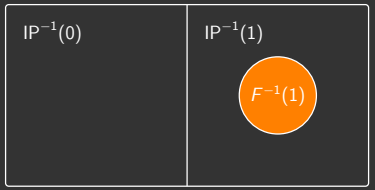

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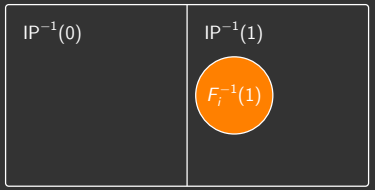

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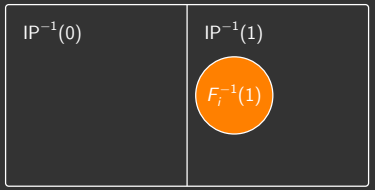

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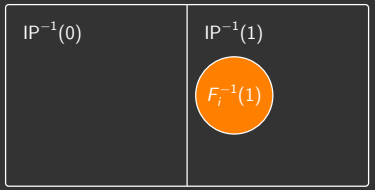

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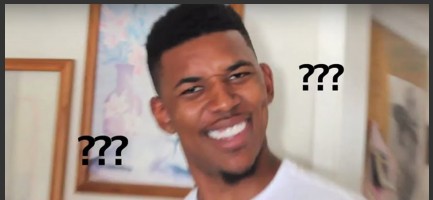

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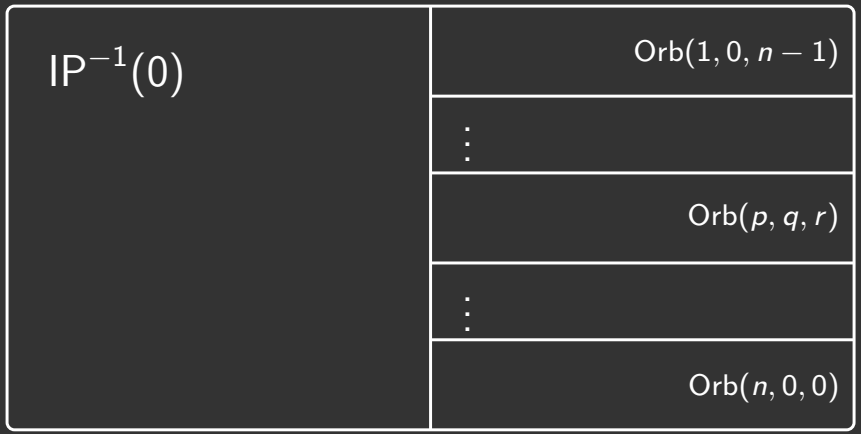

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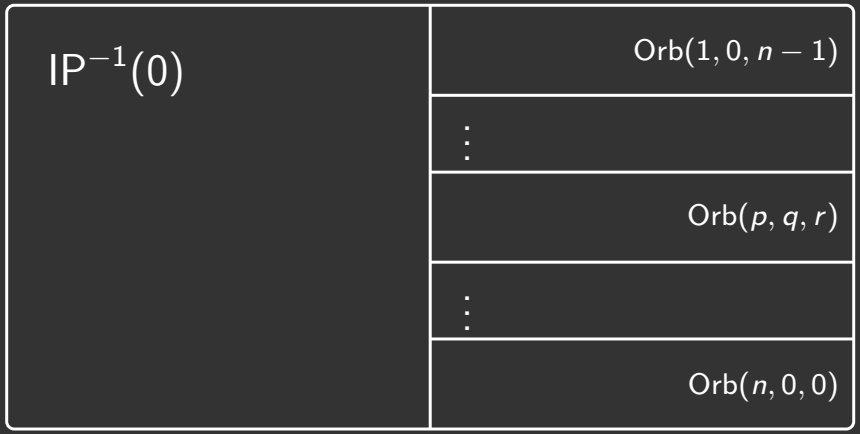

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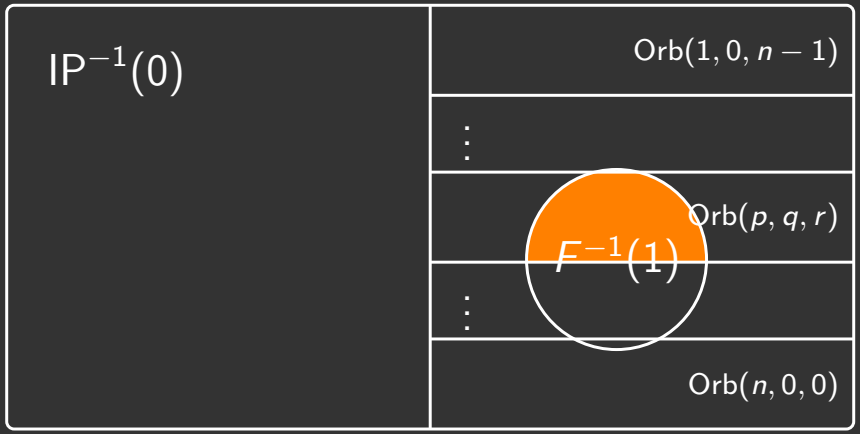

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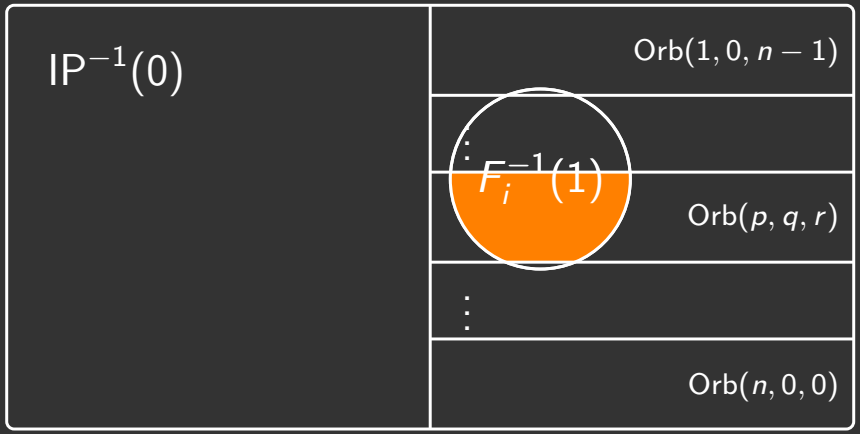

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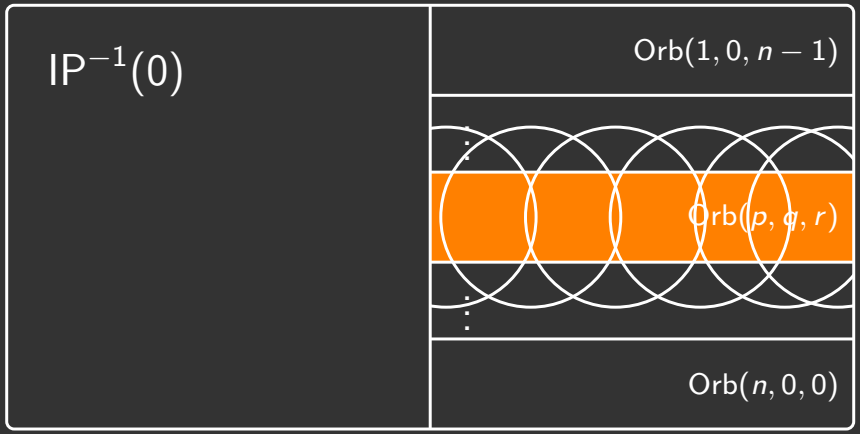

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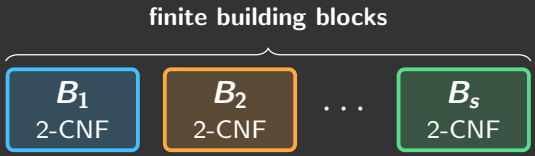

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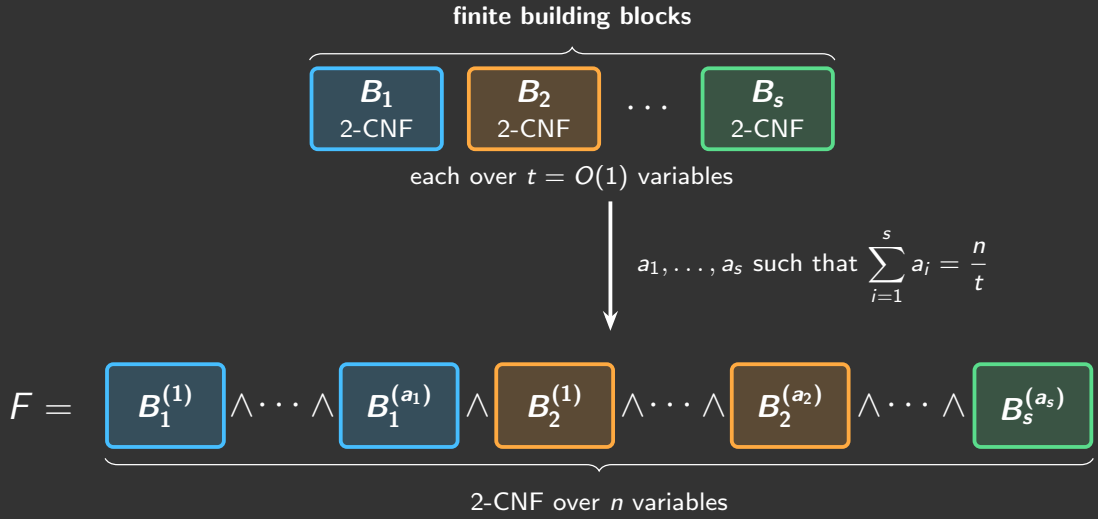

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# Constructing Finite Set of Building Blocks

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With **3** building blocks over $t = 4$ variables, and $n = 200$, the circuit size was $(1.8)^n$

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- **Crucial Observation**: If $F$ is obtained by taking **disjoint copies** of $F_1$ and $F_2$, then

$$\boldsymbol{G(x, y, z) = G_1(x, y, z) \cdot G_2(x, y, z)}$$

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- And that circuit size is $\max_{p,q,r} \frac{|\text{Orb}(p,q,r)|}{\text{coefficient of } x^p y^q z^r} \leq (1.8)^n$

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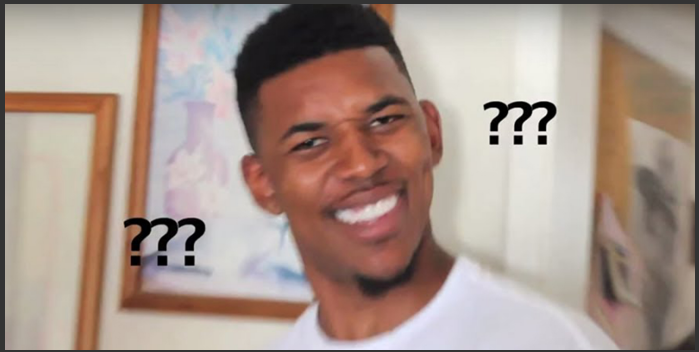

## Regions of Orbits

Let $\mathcal{R}_i(n) \subseteq \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ be $(p, q, r) \in \mathbb{N}$ such that $p + q + r = n$ and

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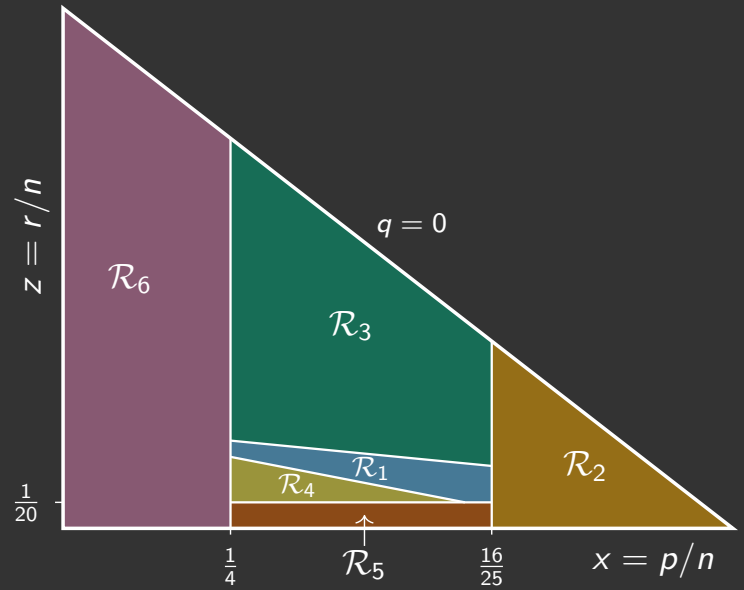
$$\mathcal{R}_5(n) : \frac{1}{2}n - \frac{25}{32}p \geq 0, -\frac{1}{20}n + r \leq 0$$

$$\mathcal{R}_6(n) : \frac{1}{4}n - p \geq 0$$

## Six regions




$$x = \frac{p}{n},$$
$$z = \frac{r}{n},$$
$$\frac{q}{n} = 1 - x - z$$

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Solved earlier using **Analytical Combinatorics** :)

## Combining Building Blocks for Orbits in Region 2/5/6

Given $p, q, r \in \mathcal{R}_i(n)$, find $A, B, C \in \mathbb{N}$ with $A + B + C = n/2$ s.t. for $G = (G_{\text{Matching}})^A (G_{\text{2Imp}})^B (G_{\text{NAND}})^{2C}$,
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**Simple** constructions and **straightforward** calculation

## Combining Building Blocks for Orbits in Region 3/4

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- Used IbexOpt, **optimization tool** part of interval arithmetic library IBEX in C++

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- Used **generating functions**, **analytical combinatorics**, **direct analysis**, and **Ibex optimization solver** to optimally **combine** building blocks


$\Sigma\Pi\Sigma_2$ Circuit for IP
Brute Force
Dynamic Programming
Generating Functions
Analytical Combinatorics
Direct Analysis
Ibex Optimization Solver

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## Degree 2 polynomial with large $\Sigma\Pi\Sigma_k$ complexity

- Construct **explicit** degree **2** polynomial with $\Sigma\Pi\Sigma_k$ complexity $2^{n-o(n)}$
- Open even for $k = 2$
- **Existentially**, shown by Impagliazzo-Paturi-Zane'01

# Thank you