All Polynomial Generators Preserve Distance with Mutual Correlated Agreement

Sarah Bordage, Alessandro Chiesa, **IM**, Ziyi Guan

Why Do We Care?

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PCP/IOP soundness

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Low-degree testing → PCP/IOP soundness

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Proximity Gaps → Low-degree testing → PCP/IOP soundness

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(δ, ϵ)-proximity gaps for $\mathsf{C} \subset 2^{\Sigma^n}$, $\mathsf{P} \subset \Sigma^n$ under distance Δ:
$\forall E \in \mathsf{C}$ either

1. $\Pr_{s \in E}[\Delta(s, \mathsf{P}) \leq \delta] = 1$

2. $\Pr_{s \in E}[\Delta(s, \mathsf{P}) \leq \delta] \leq \epsilon$

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In this work: consider spans of polynomial proximity generators, $\mathsf{P} = \mathscr{C}$ linear code, under the Hamming distance.

Distance Preservation

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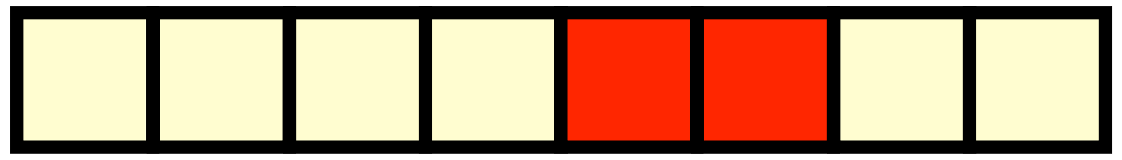
$\mathbf{u}_1$  $\delta_1 := \Delta(u_1, \mathscr{C})$

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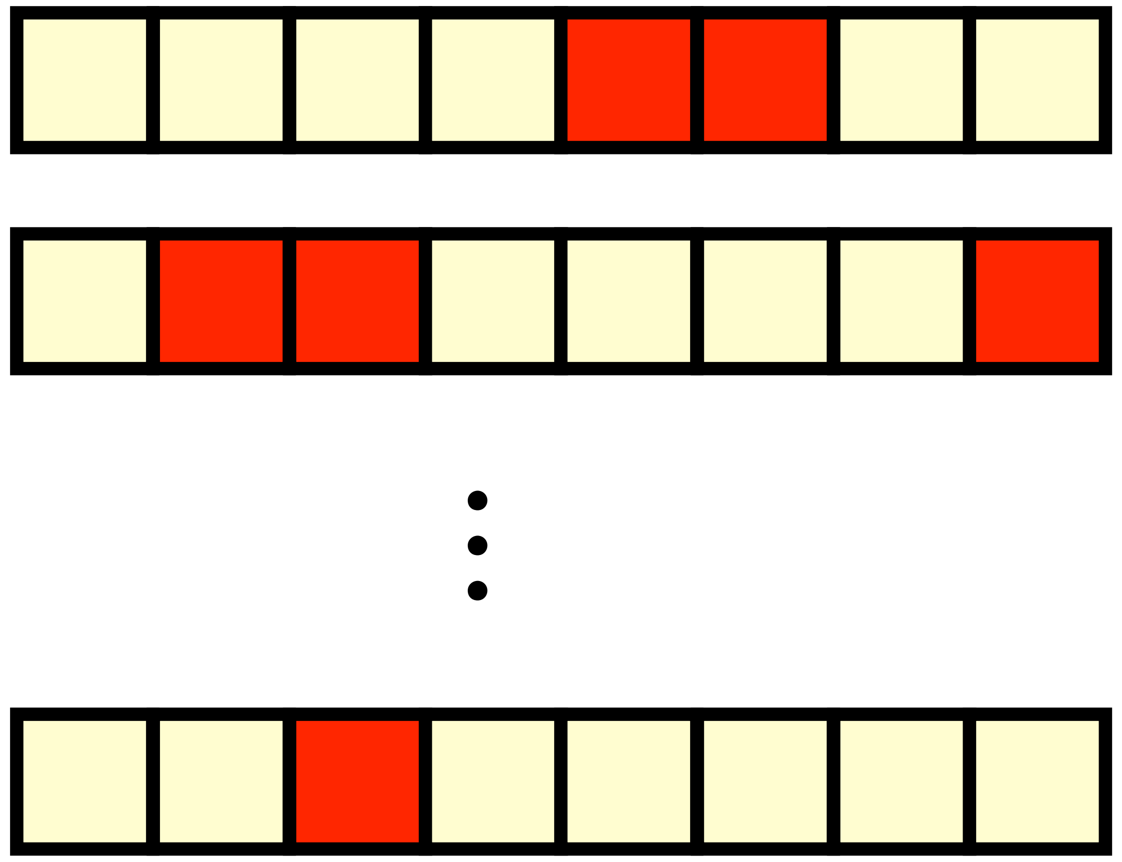

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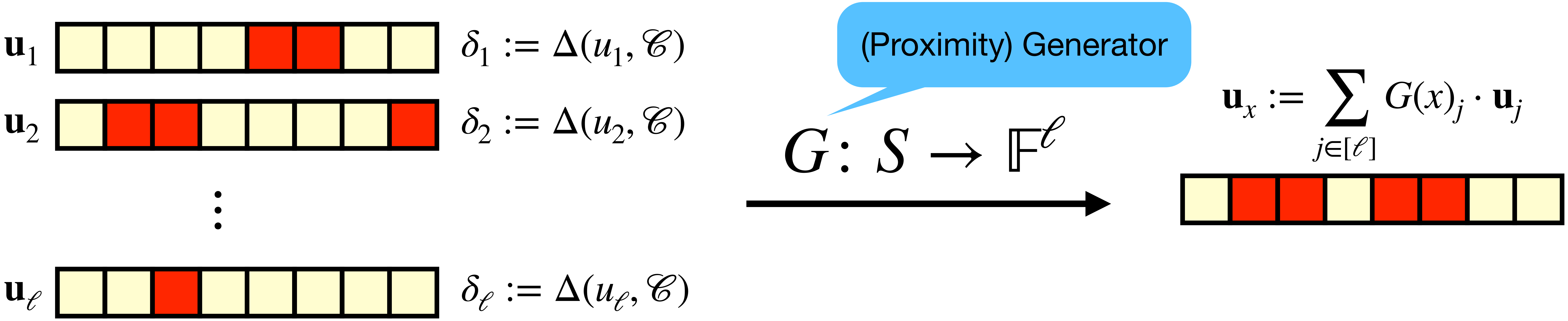

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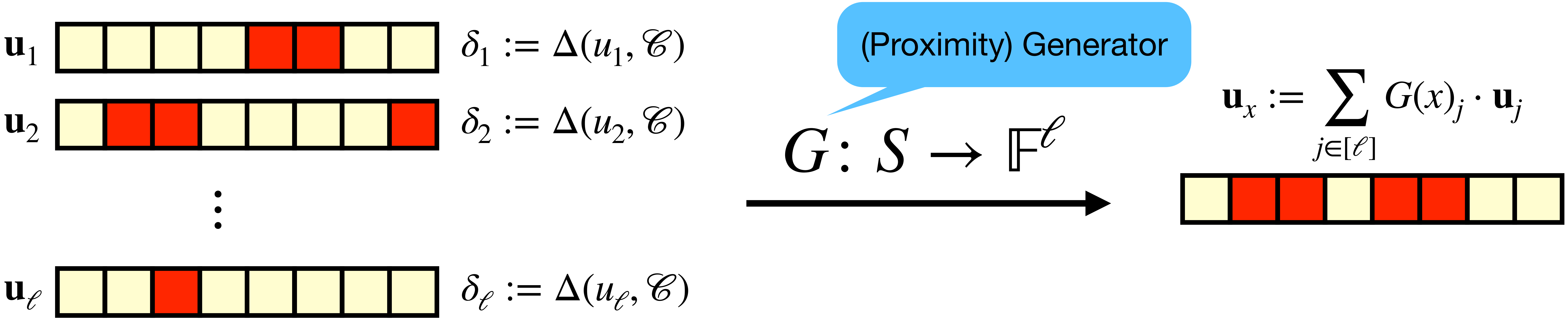

We want $\Pr_{x \in S}\left[\Delta(\mathbf{u}_x, \mathscr{C}) < \max_j \delta_j\right]$ to be small

Preserving Max Distance is not Enough

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[RVW13]

When $G(x_1, \ldots, x_\ell) = (x_1, \ldots, x_\ell)$,

$$\Pr_{\vec{x} \in \mathbb{F}^\ell}\left[\Delta(\mathbf{u}_{\vec{x}}, \mathscr{C}) < \max_j \frac{\delta_j}{2}\right] \leq \frac{1}{|\mathbb{F}|}$$

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(e.g. batch proximity testing)

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[AHIV17],[BKS18],[BGKS20]

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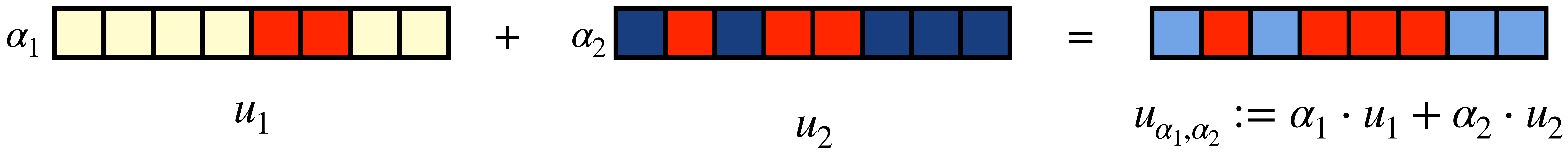

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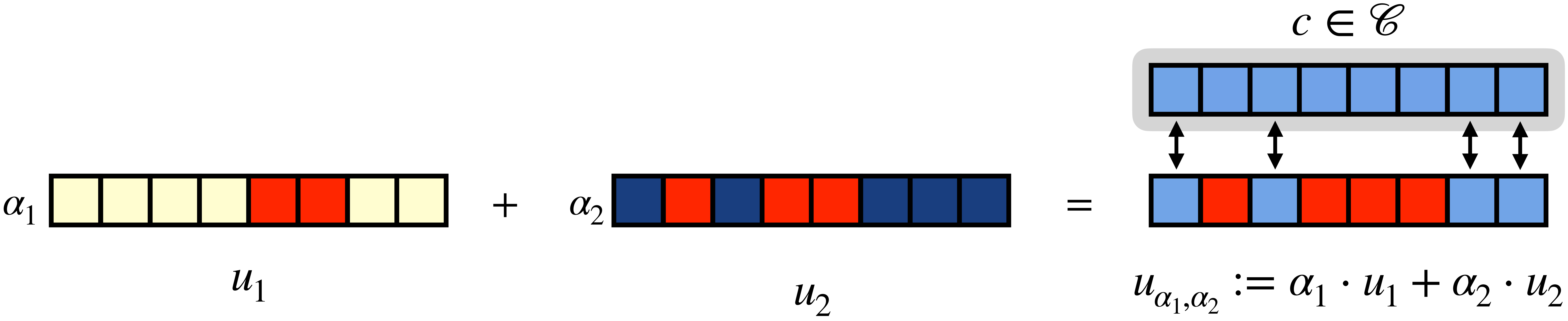

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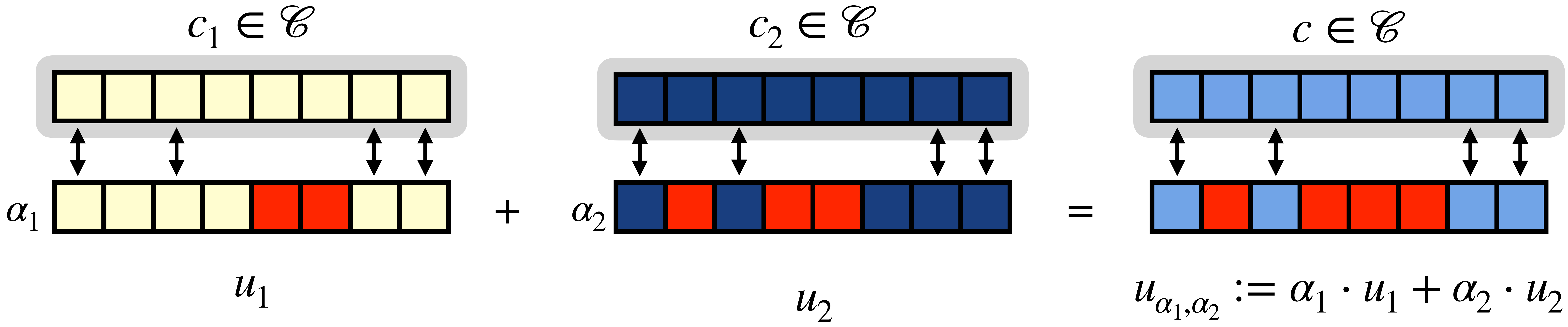

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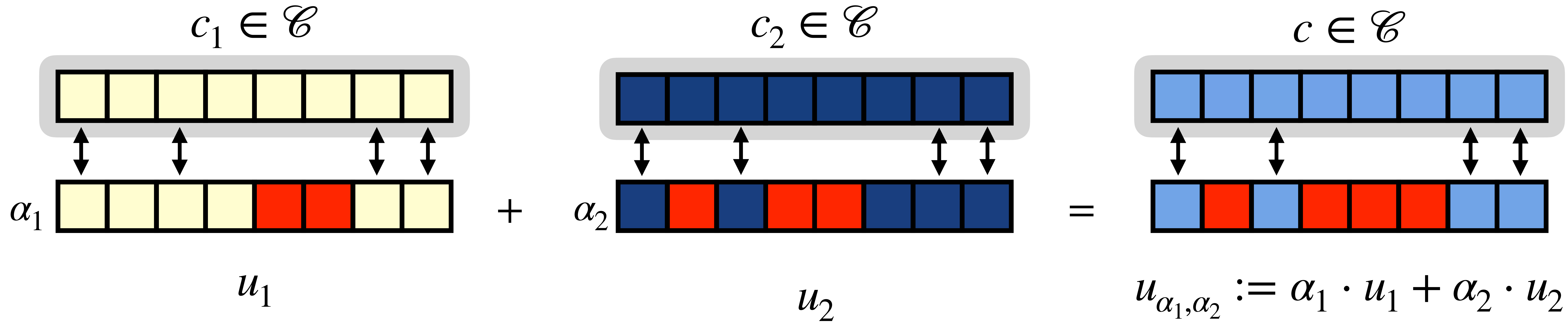

$T := \{1,3,7,8\}$ - the common agreement set

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Testing membership to the code!

MCA Makes Things Nice

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List decoding Key step in security reduction of IOPs

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List decoding

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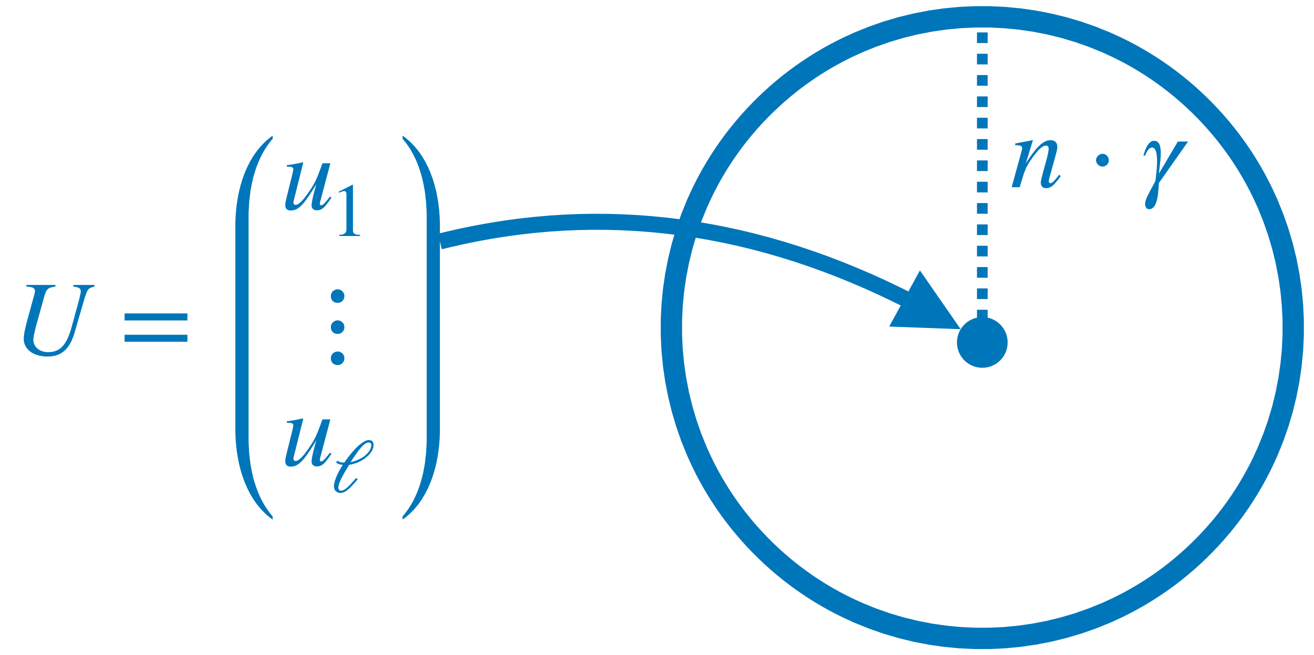

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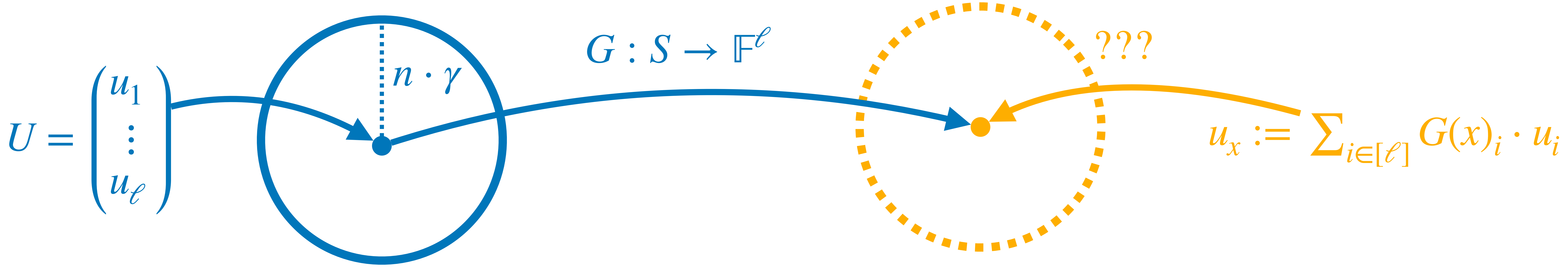

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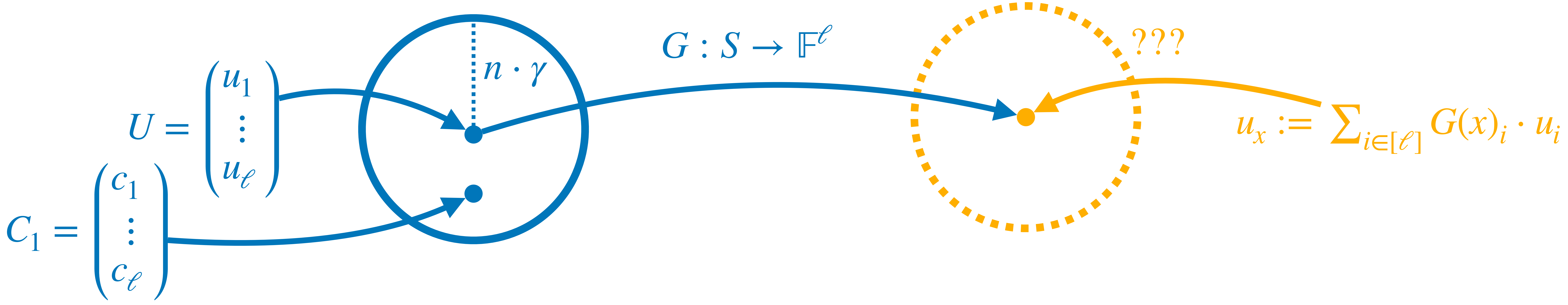

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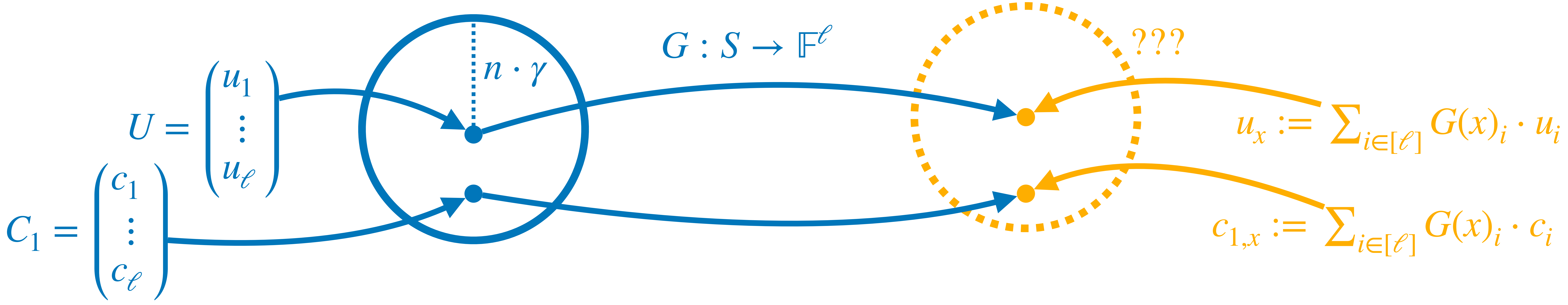

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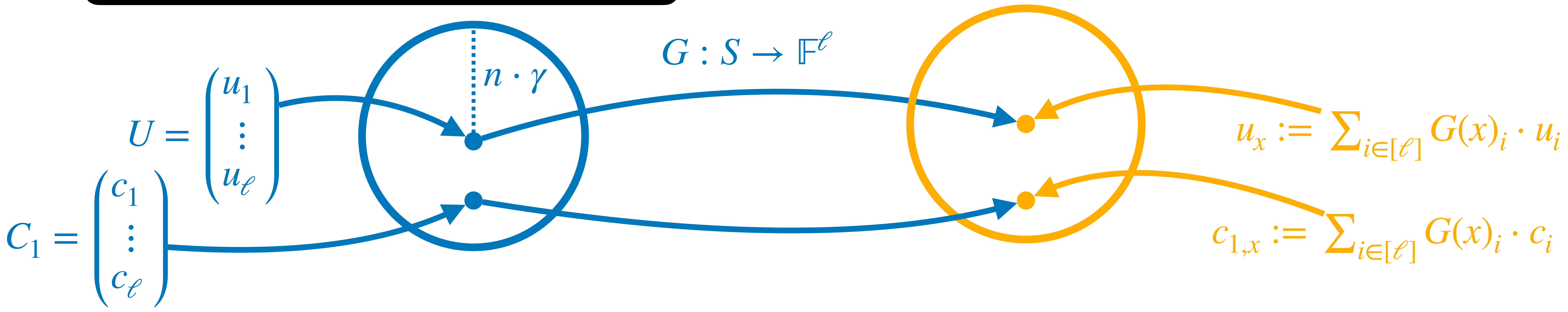

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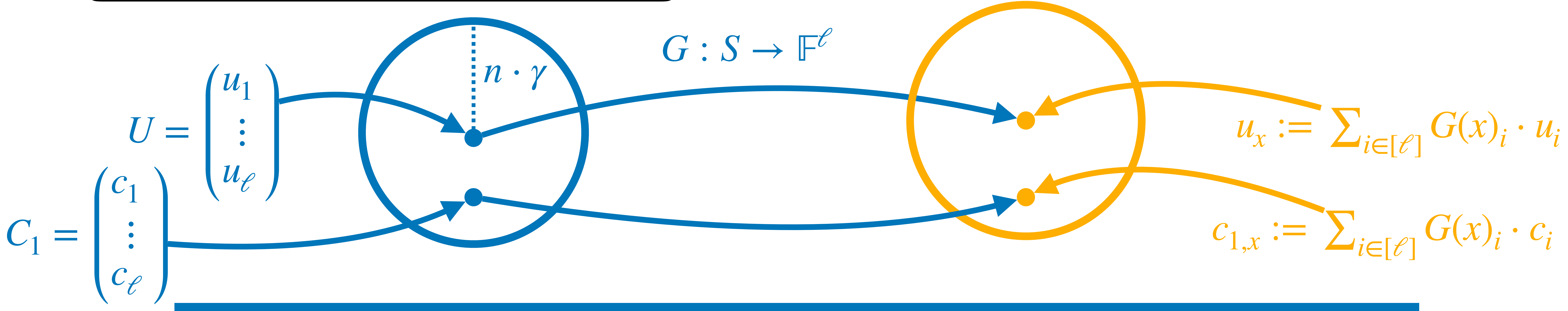

"List-decode and combine" ⇔ "Combine and list-decode"

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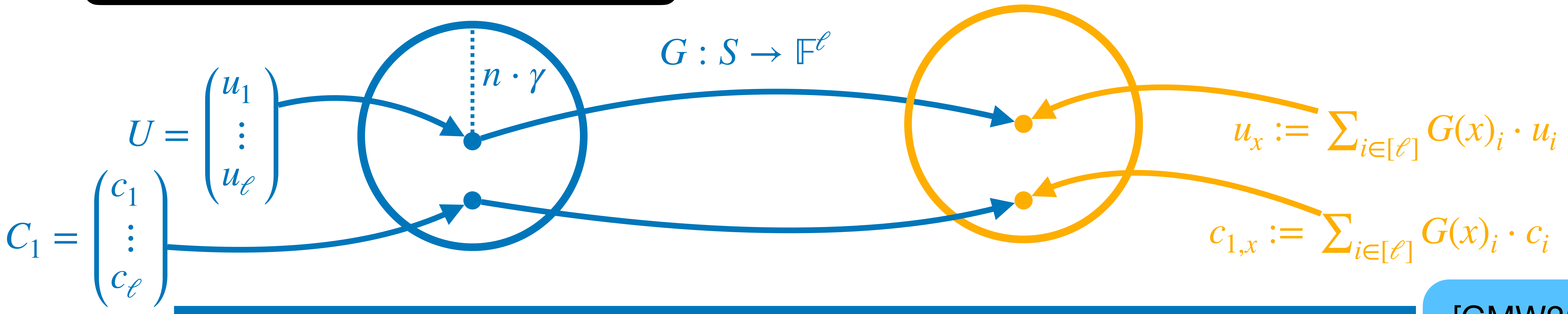

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[GMW25]

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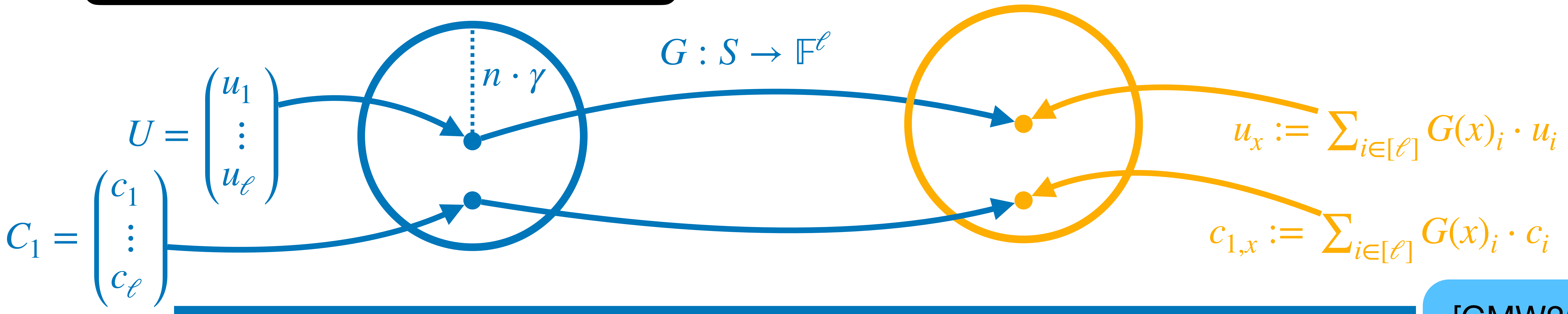

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Easier analyses, better parameters

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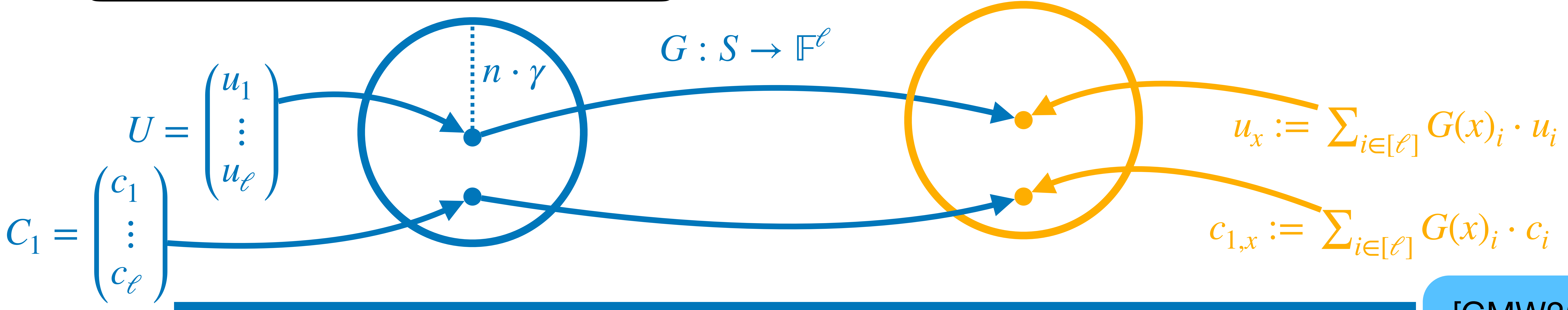

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RBR Security

Essential for Fiat—Shamir

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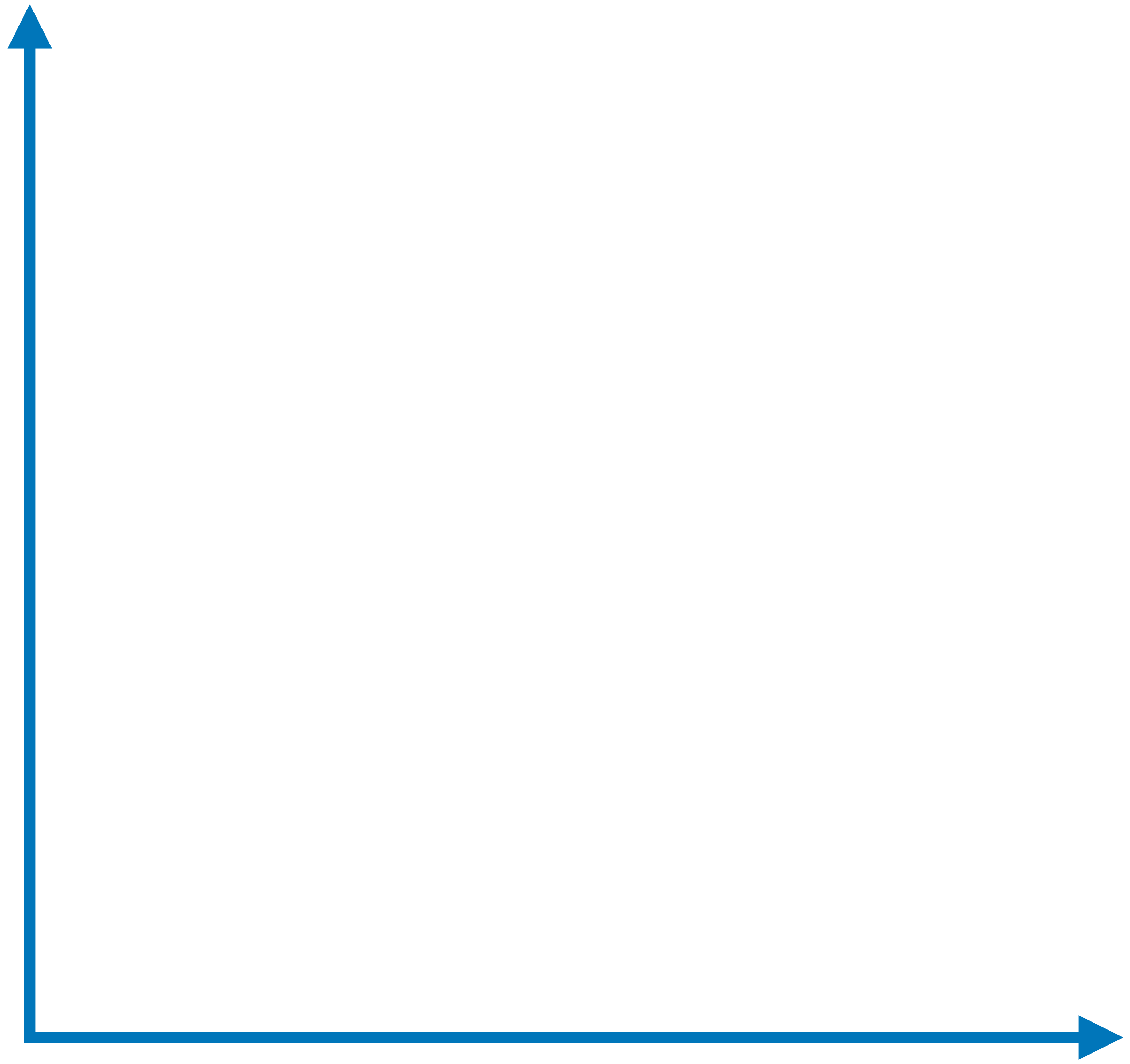

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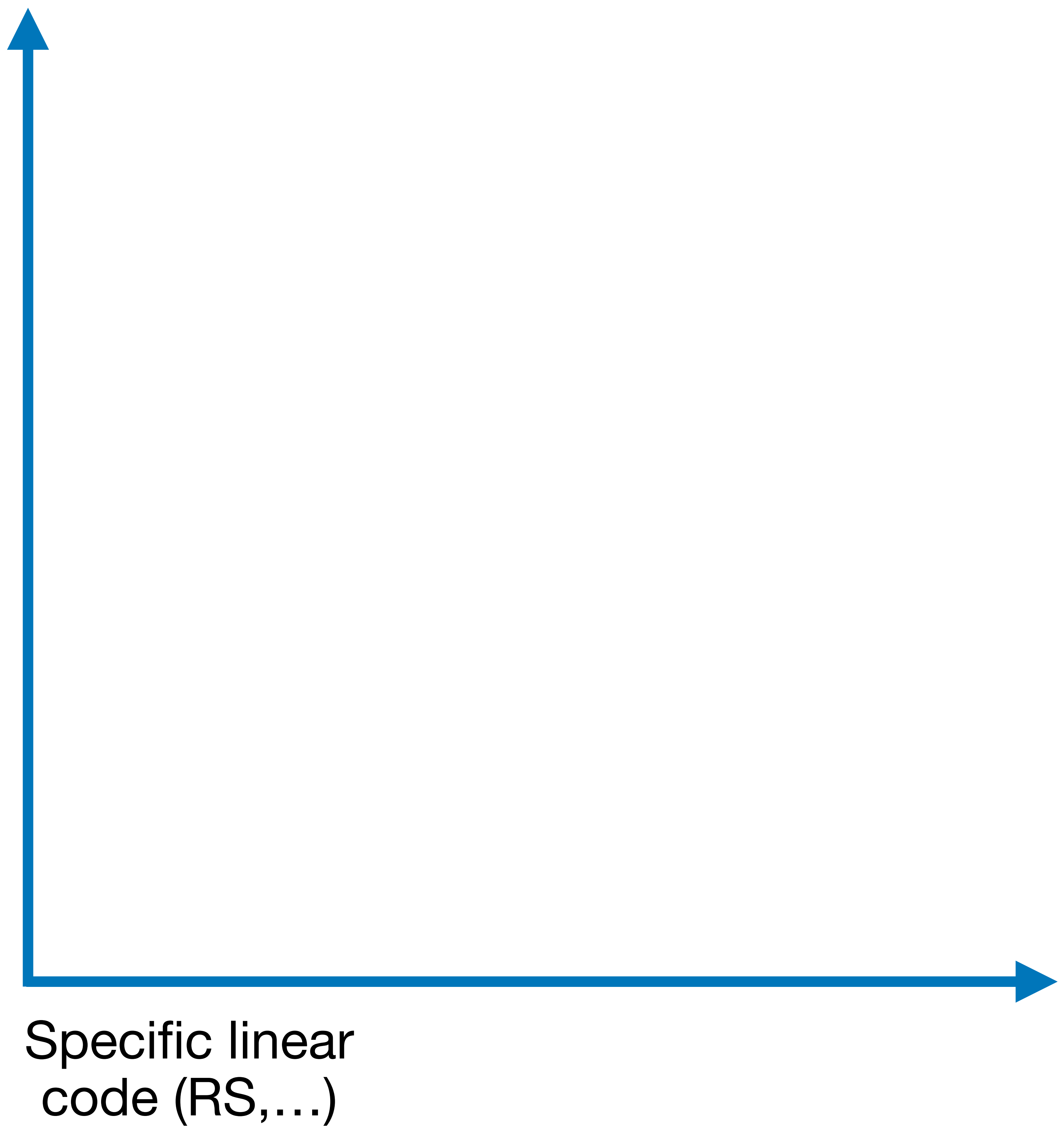

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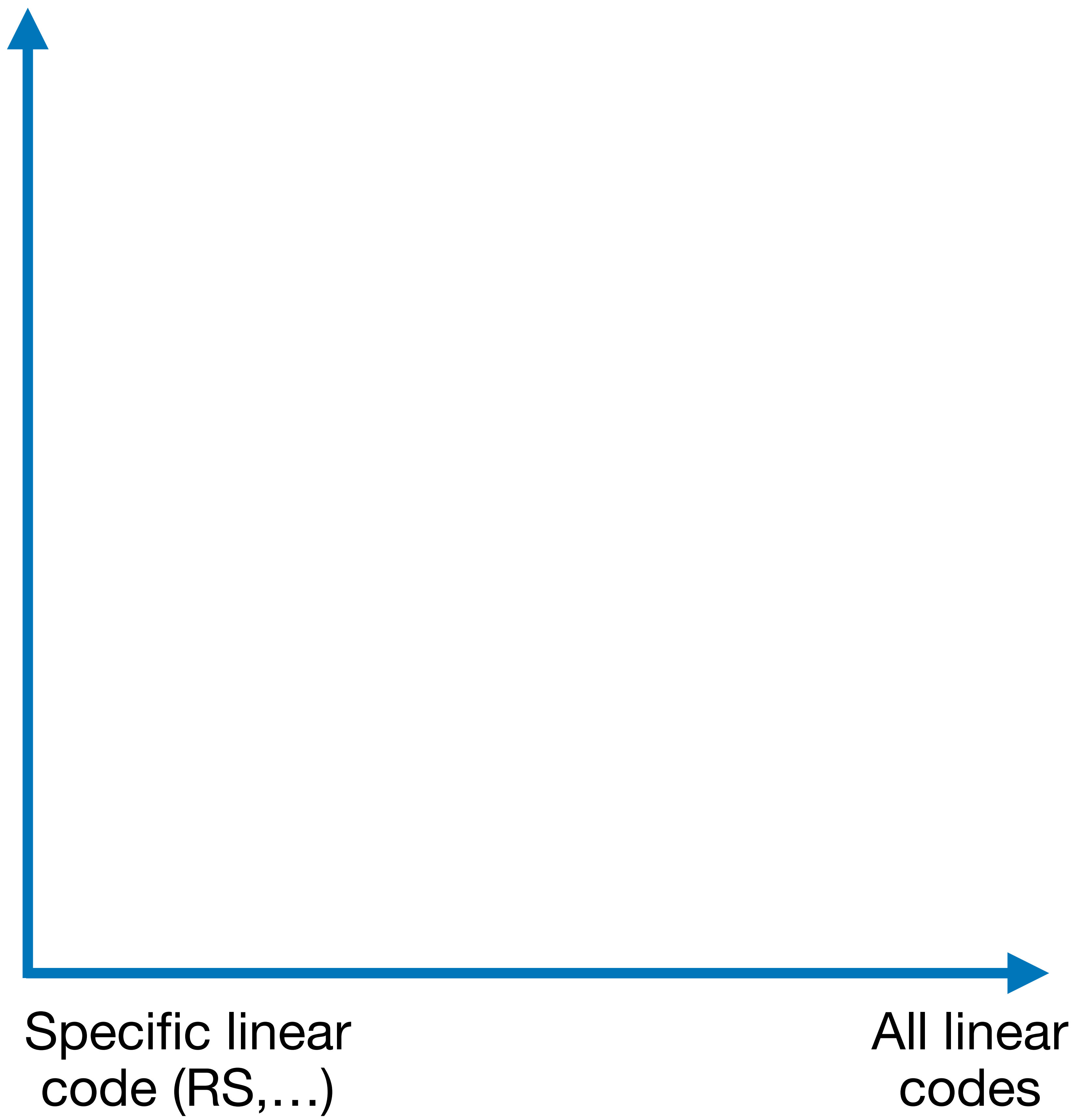

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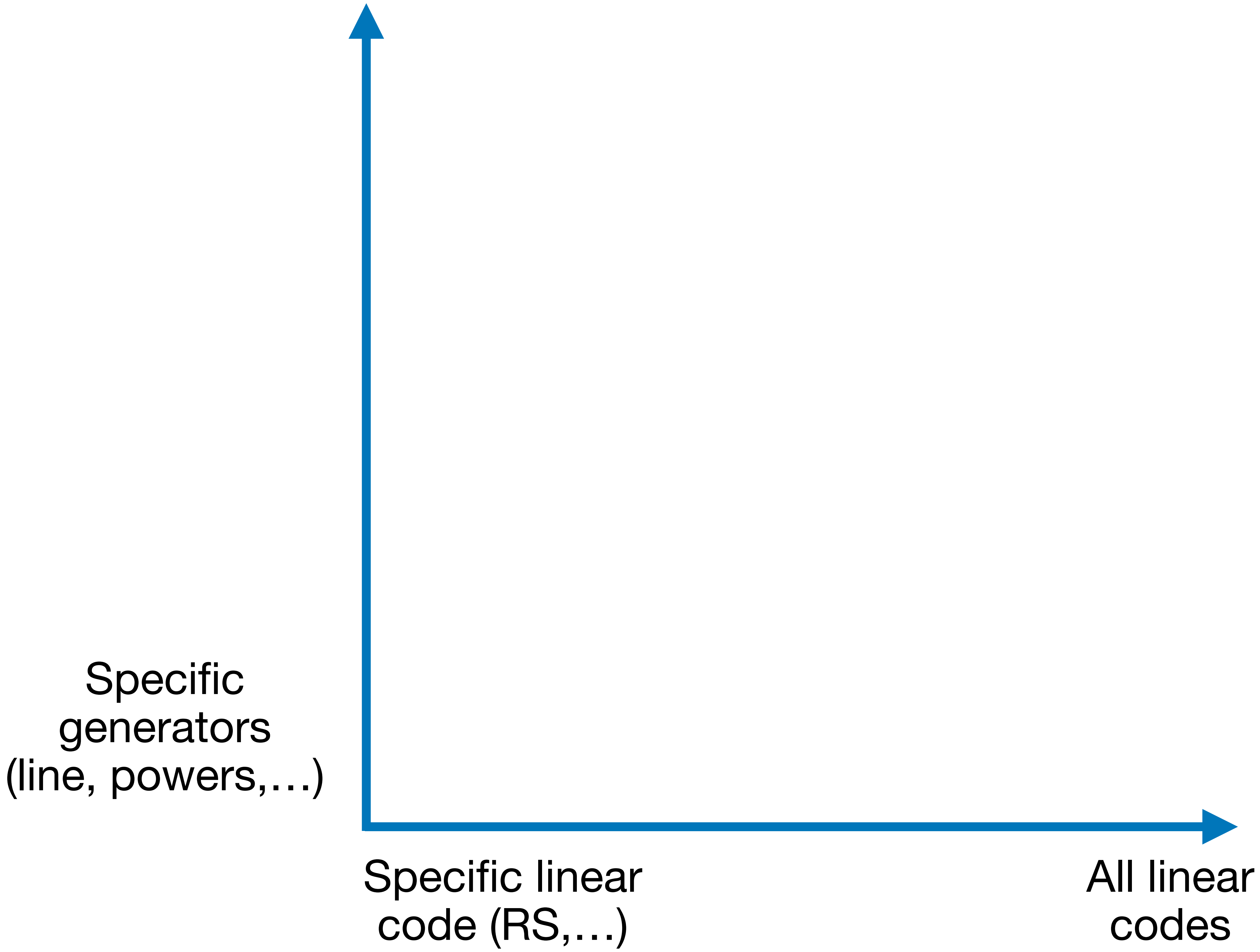

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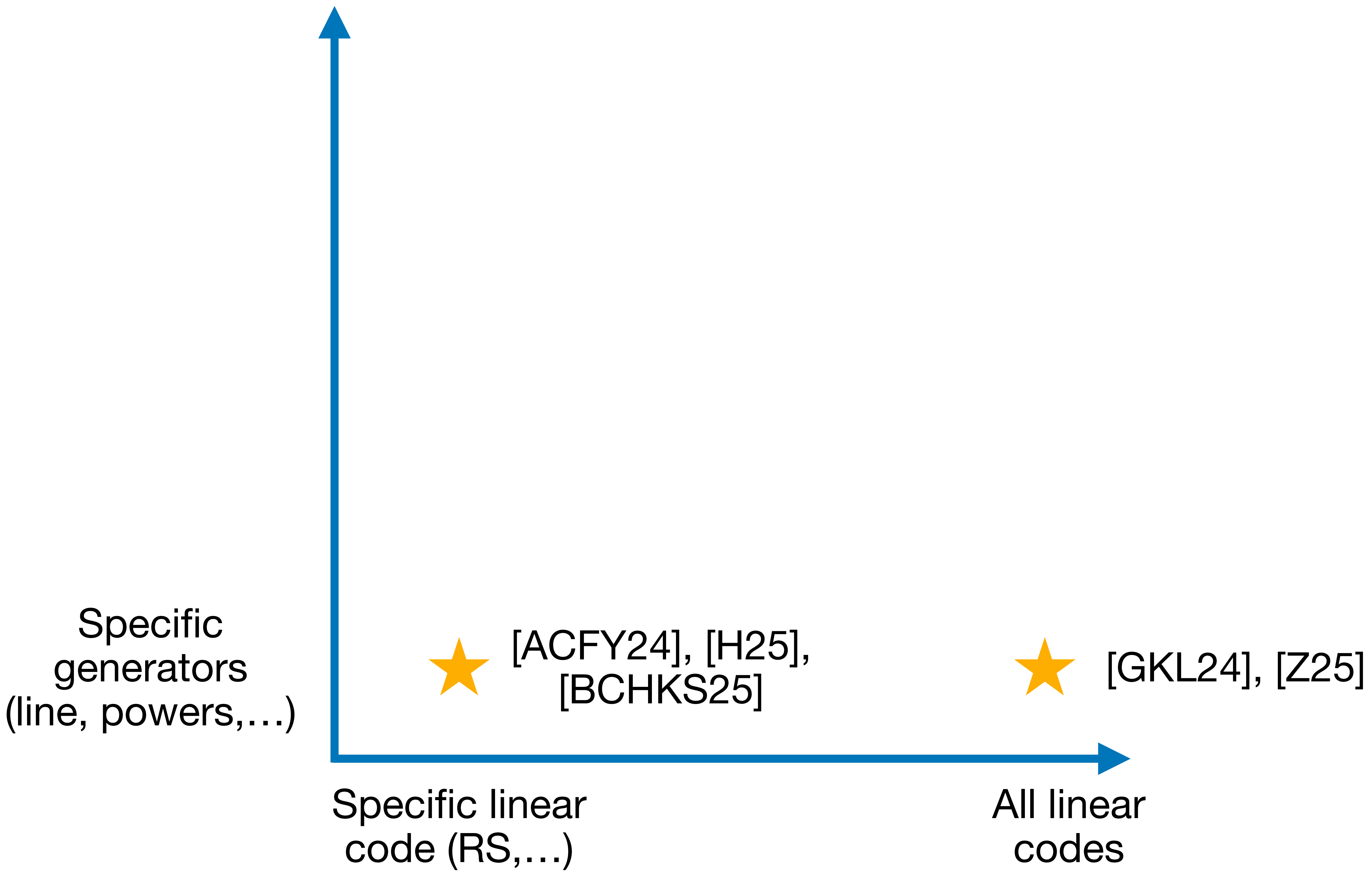

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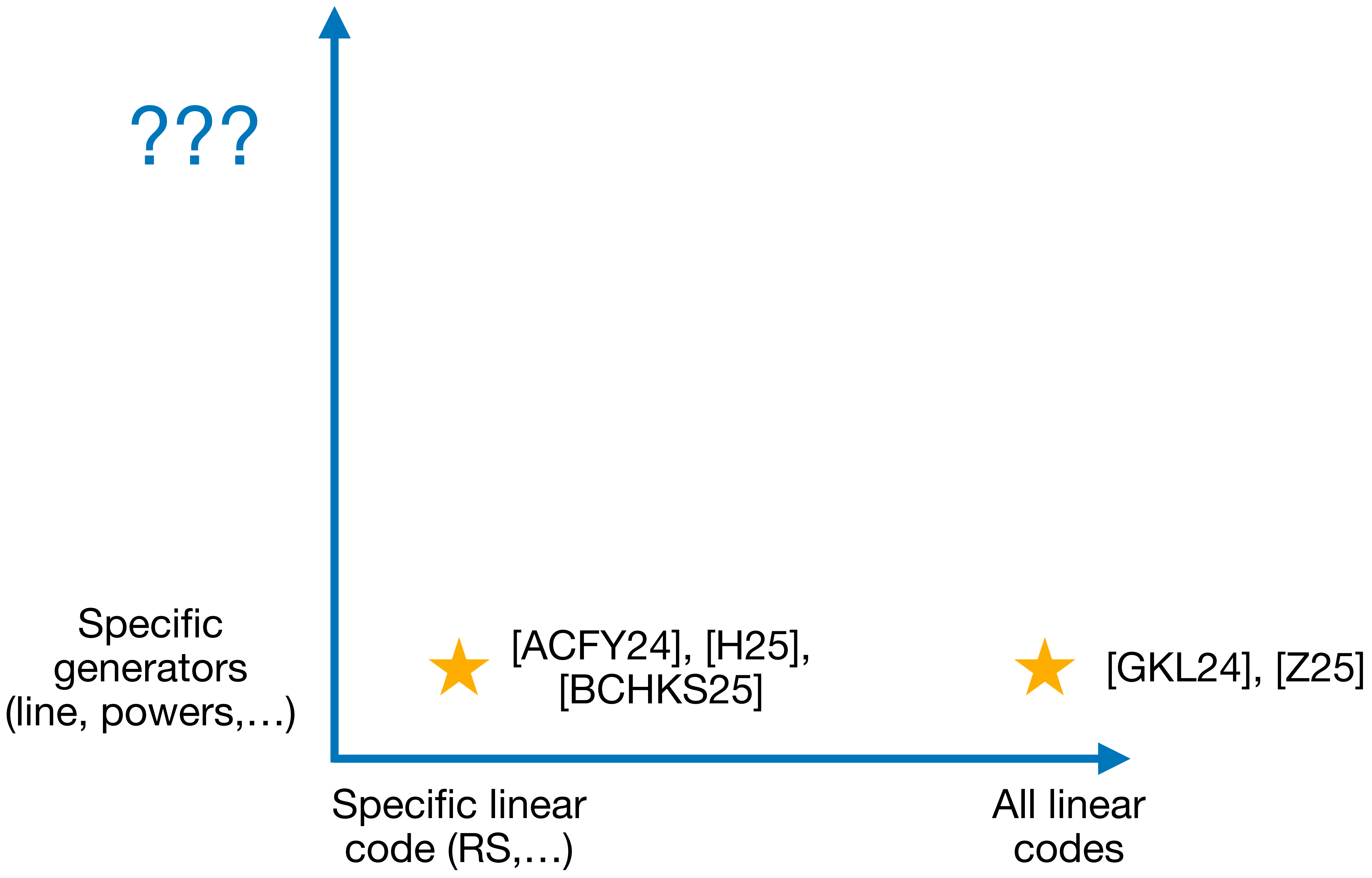

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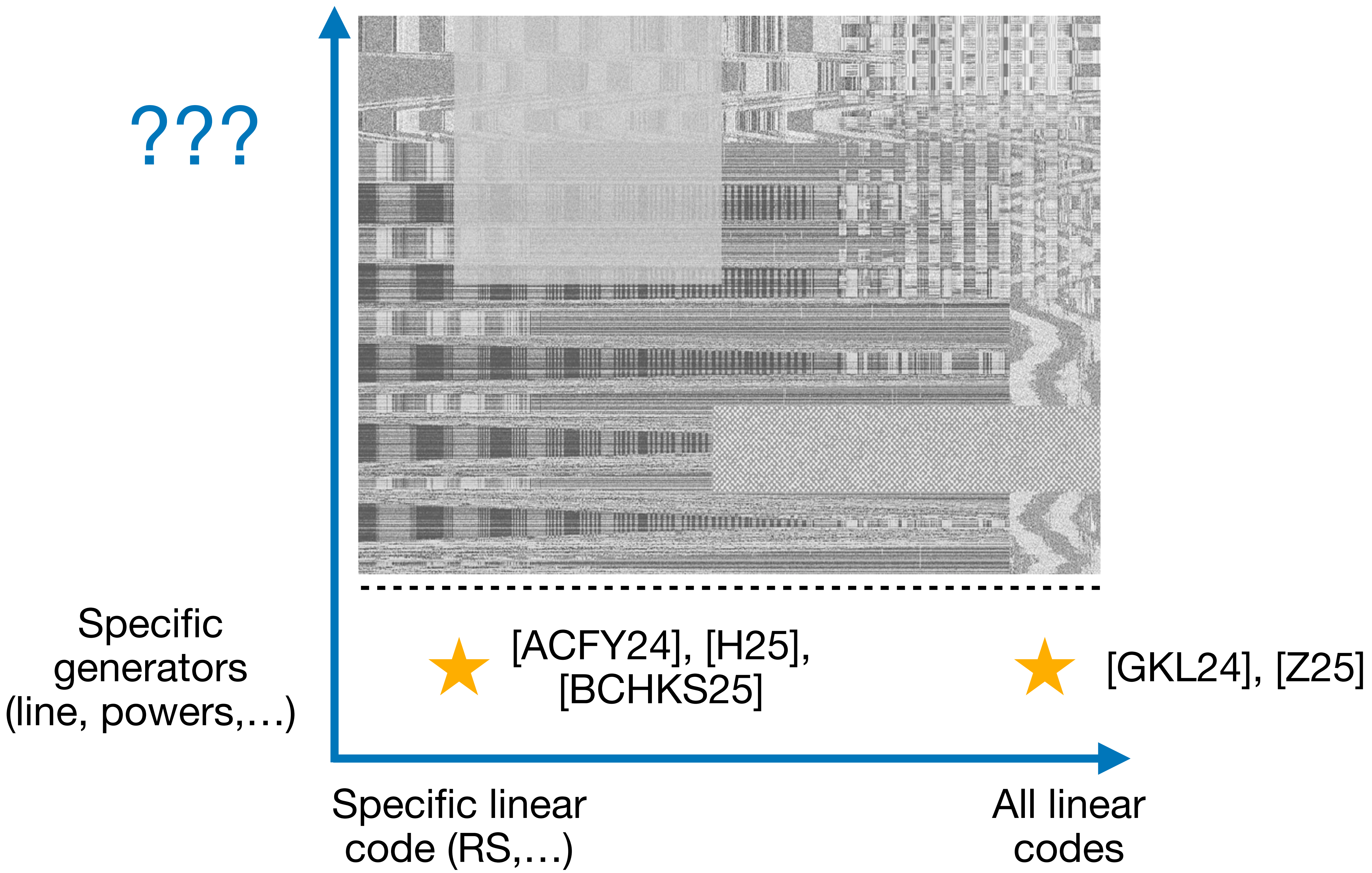

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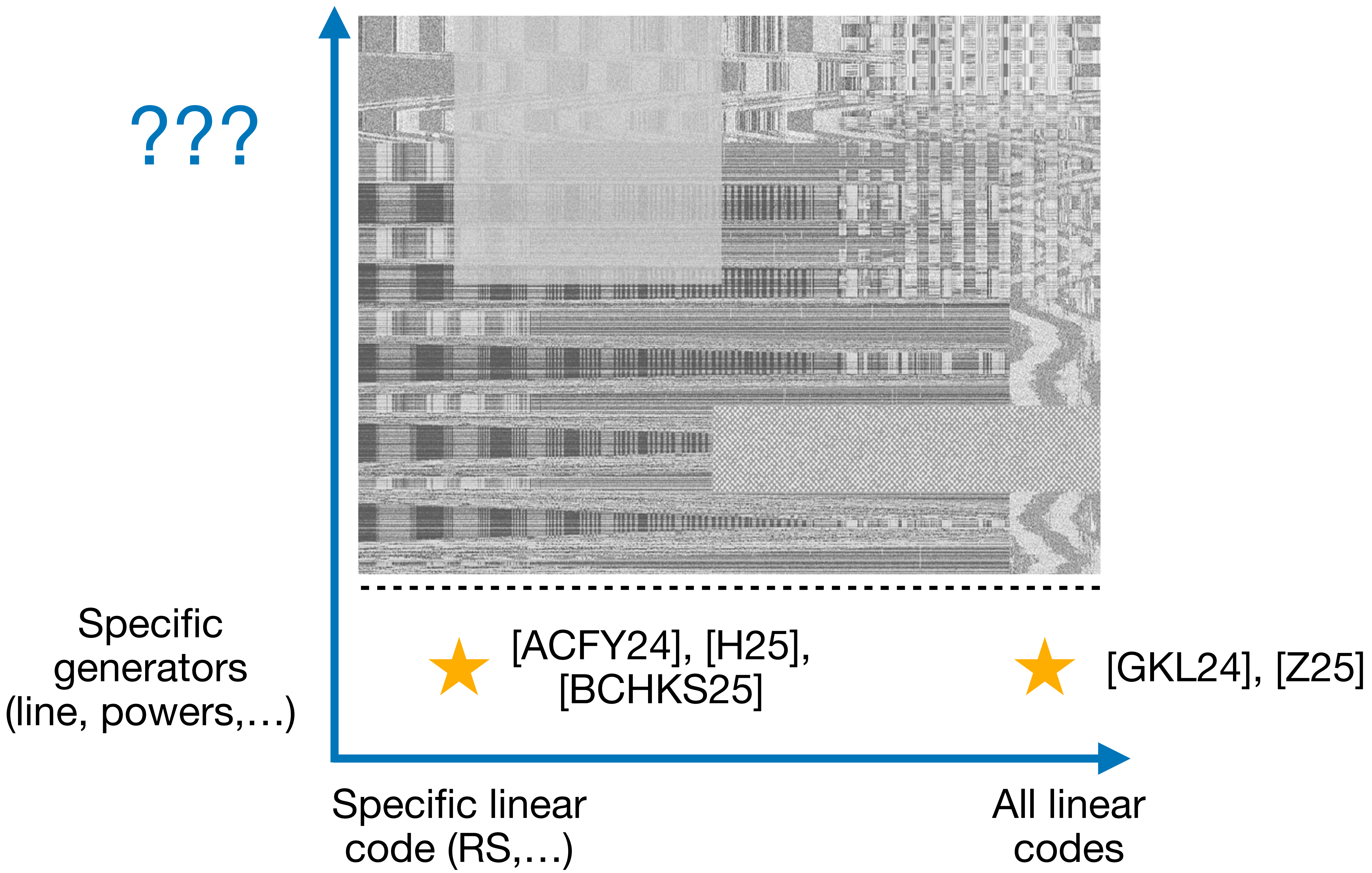

Which generators have MCA?

Our Results

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3. ***MCA Toolbox***

The new MCA landscape

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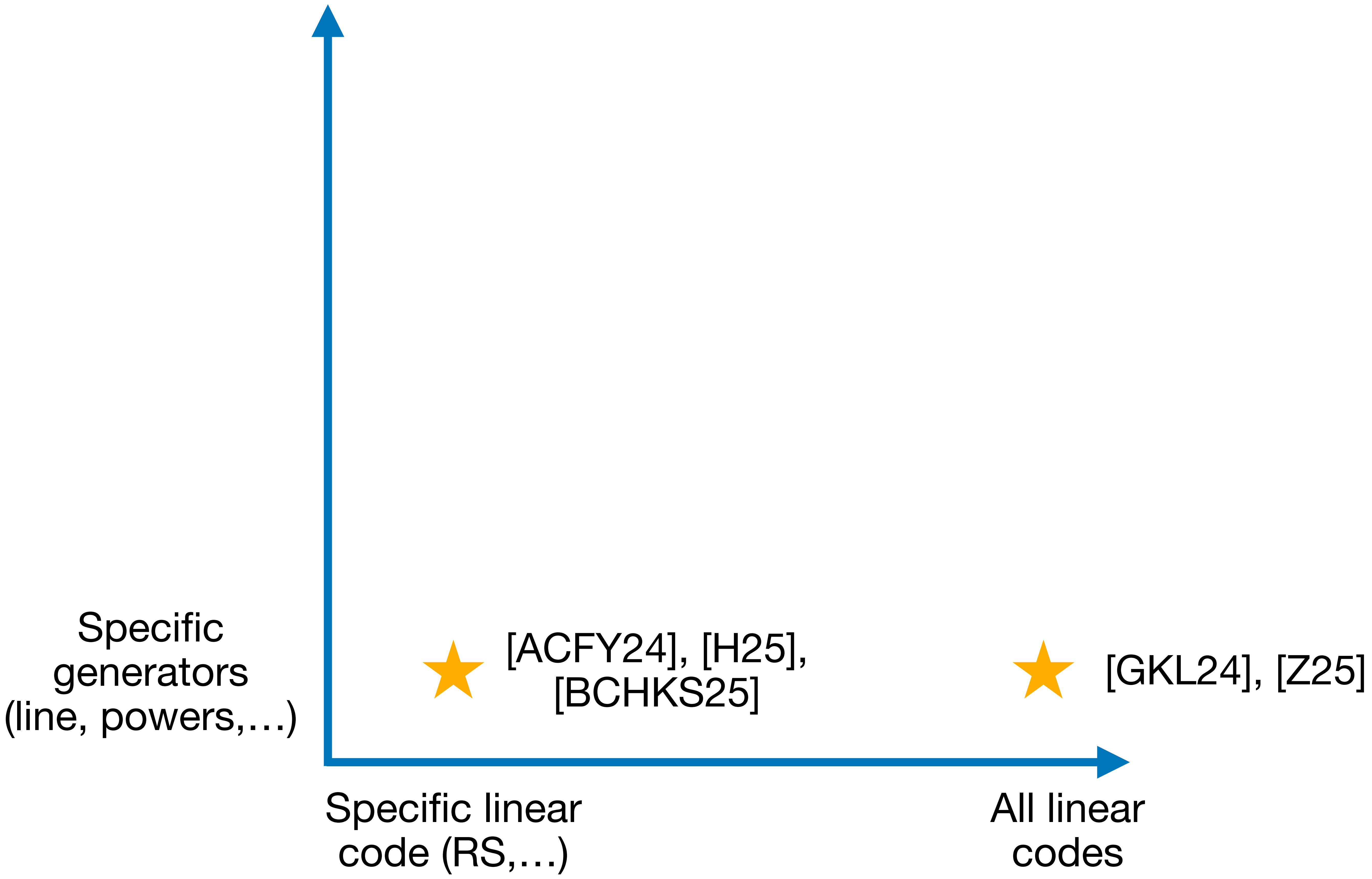

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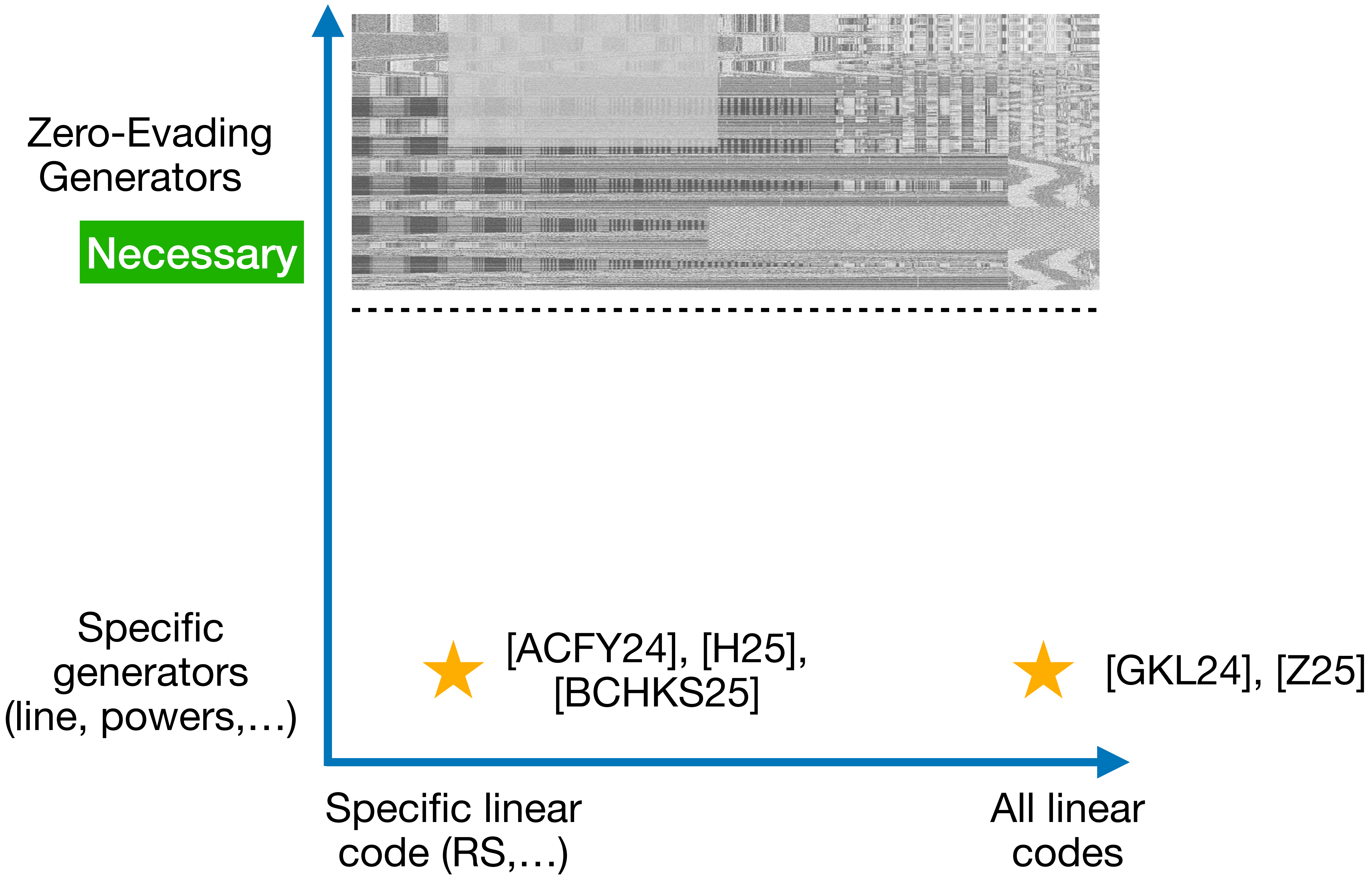

The new MCA landscape

$\overrightarrow{X} \mapsto (\overrightarrow{X}^{\mathbf{v}})_{v \in \{0,1\}^s}$

$(X_1, \ldots, X_s) \mapsto (X_1, \ldots, X_s)$

$X \mapsto (1, X, X^2, \ldots, X^d)$

$X \mapsto (1, X)$

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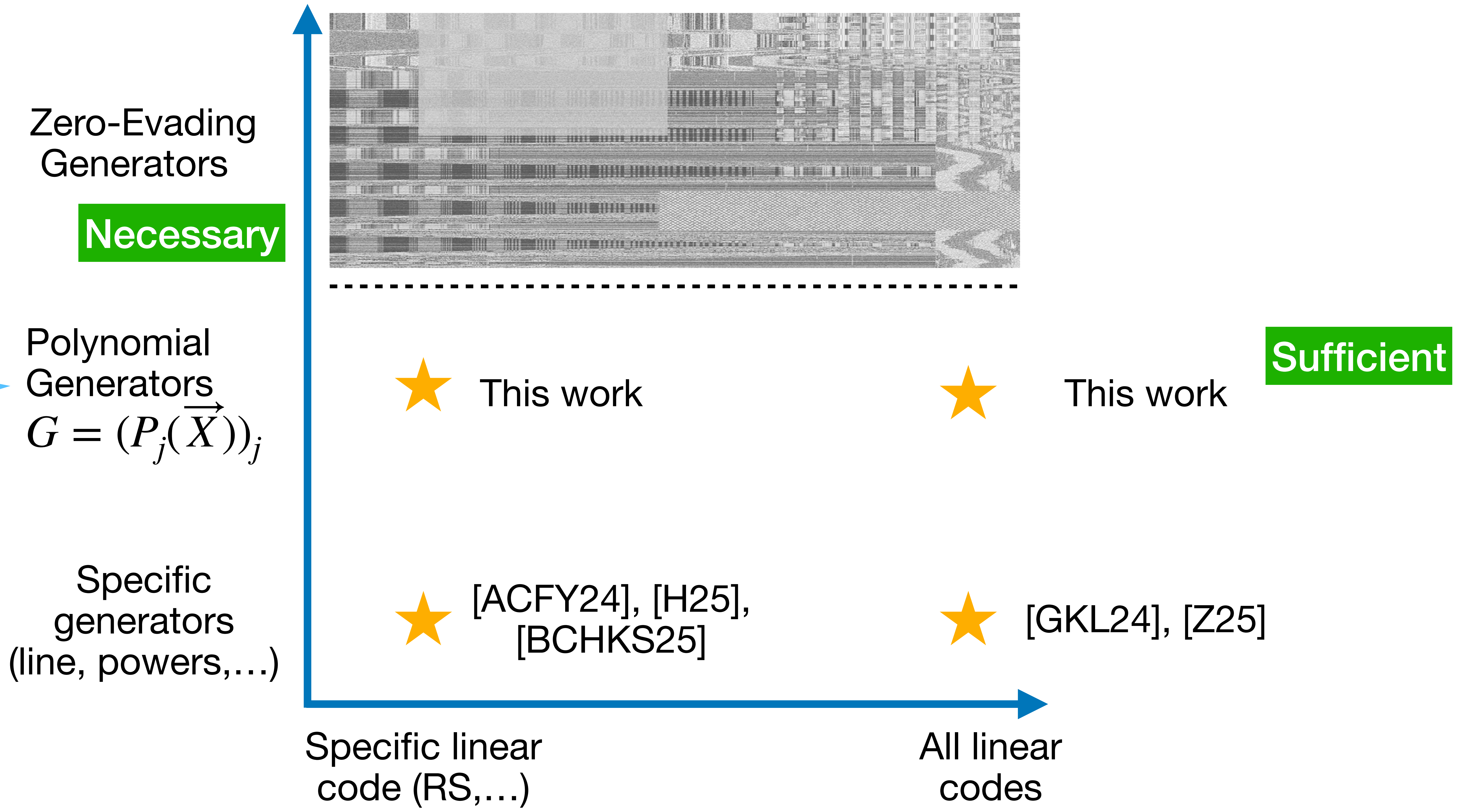

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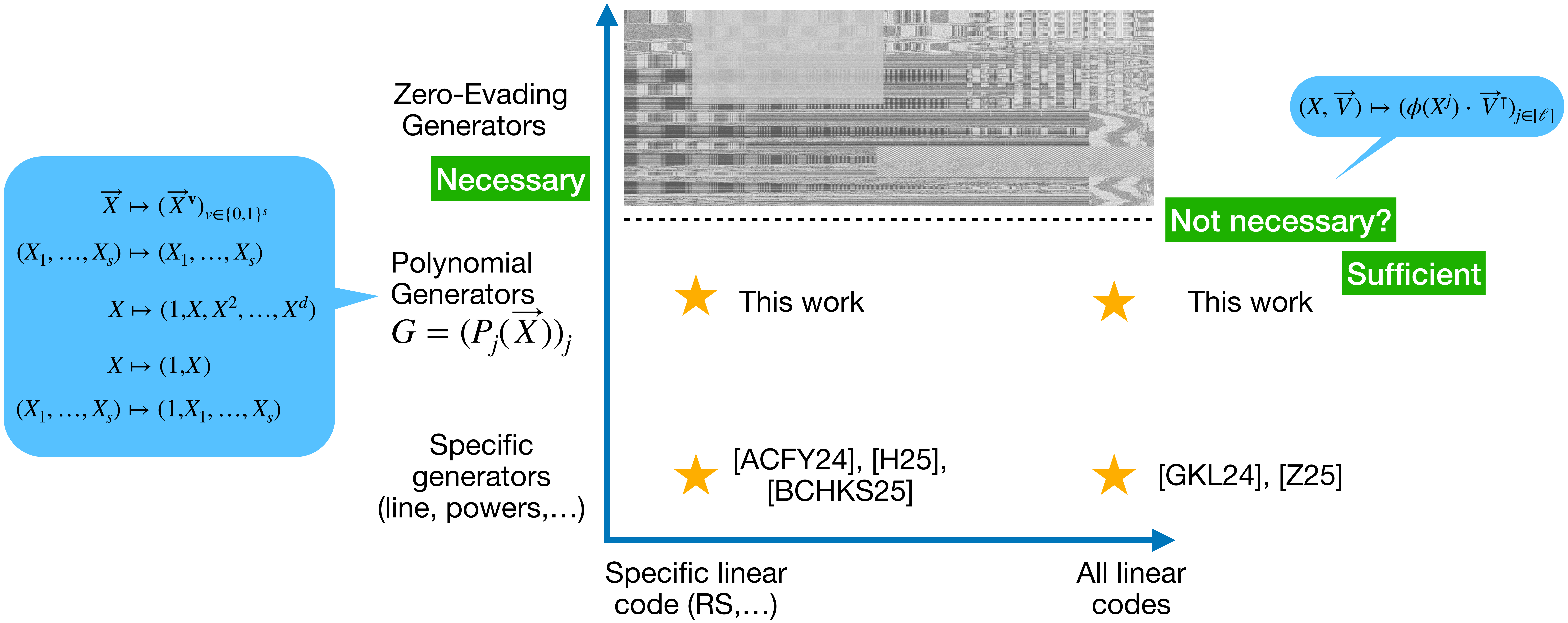

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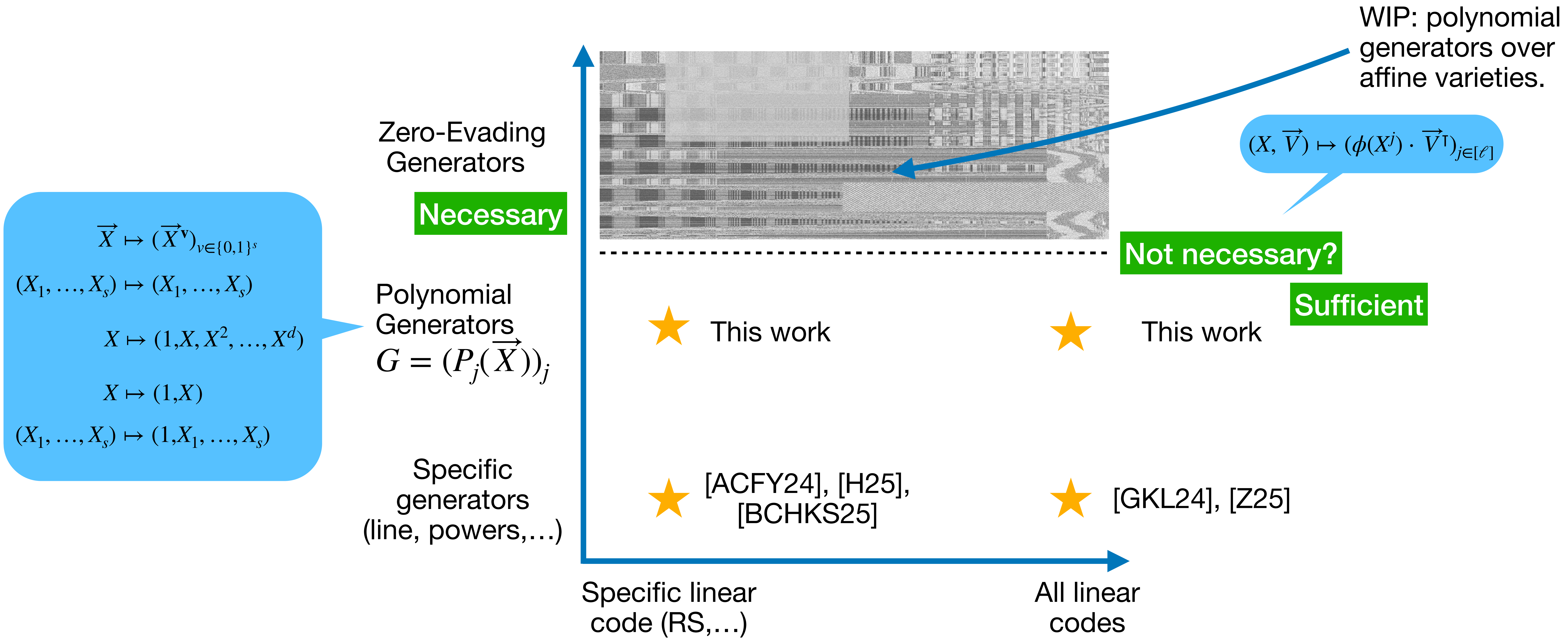
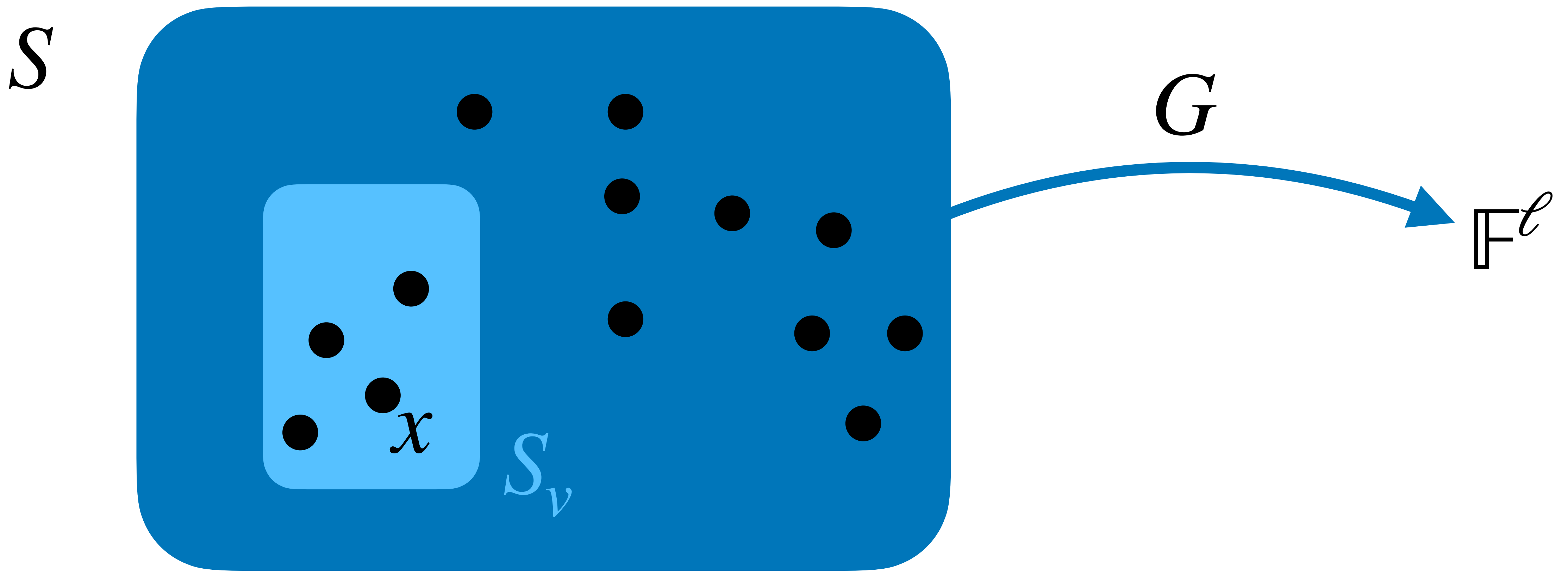


ZE is Necessary

Zero-evading is necessary for MCA: $\epsilon_{ZE} \leq \epsilon_{MCA}$



$$S_v := \{x \mid G(x) \cdot v^\top = 0\} \quad (v \neq 0)$$

$$\max_{v \neq 0} |S_v| / |S| \leq \epsilon_{ZE}$$

ZE is Necessary

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$$\epsilon_{MCA}(0) = \Pr_{x \in S}\begin{bmatrix} G(x) \cdot \mathbf{U} \in \mathscr{C} \\ \vee_{j \in [\ell]} \mathbf{u}_j \notin \mathscr{C} \end{bmatrix}$$

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proof: $\mathscr{C} \subseteq \mathbb{F}^n$ of codimension r, $M \in \mathbb{F}^{r \times n}$ parity check matrix.

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$\vec{v} = (\mathbf{w}_1[i], \ldots, \mathbf{w}_\ell[i])$
for $i \in [r]$ s.t. $\mathbf{w}_j[i] \neq 0$

$\blacksquare$

Polynomial is Sufficient

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Linearly independent over $\mathbb{F}$

Polynomial generator $G(x_1, \ldots, x_s) = (P_1(x_1, \ldots, x_s), \ldots, P_\ell(x_1, \ldots, x_s))$

All polynomial generators have MCA for every linear code

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Pf sketch:

- Write $P_i(X_1, \ldots, X_s) = \sum_{\alpha_1,\ldots,\alpha_s} c_{\alpha_1,\ldots,\alpha_s} \cdot \prod_i X_i^{\alpha_i}$
- Have $G(X_1, \ldots, X_s) = \mathbf{A} \cdot \bigotimes_i (1, X_i, \ldots, X_i^{d_i})$

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Prove both transformations interact "nicely" with MCA: toolbox for MCA

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Can specialize to RS codes and obtain MCA up to Johnson bound for all polynomial generators

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Lower Bounds for MCA

$G(x_1, \ldots, x_s) = (P_1(x_1, \ldots, x_s), \ldots, P_\ell(x_1, \ldots, x_s))$, and $d_i := \max_j \{\deg_{X_i}(P_j)\}$

$$\epsilon_{MCA}(\gamma) = \max\{n \cdot \gamma, 1\} \cdot \sum_i \frac{d_i}{|\mathbb{F}|} \quad \text{if } \gamma < \min_i\{\frac{\delta_{\mathscr{C}}}{d_i + 2}\}$$

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Matching lower bound: $\exists$ code s.t. when $\gamma \leq \delta_{\mathscr{C}}/2$, the MCA error of G_d is $\geq n \cdot \gamma \cdot d/|\mathbb{F}|$.

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Can we close the gap?

MCA Toolbox

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Linear transformation

$$G : S \to \mathbb{F}^\ell$$

$$G' := \mathbf{A} \cdot G$$

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$$(x_1, \ldots, x_s) \mapsto (1, x_1, \ldots, x_s) \approx_{\mathbf{MCA}} x \mapsto (1, x)$$

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Matrix-vector product

Can show the generator from [AGHP92] has good MCA error

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2. over $\mathbb{F}_{q^k}$, $\phi(x^i)_j = \mathrm{tr}(\beta_j x^i) = \sum_{m=0}^{k-1} (\beta_j x^i)^{q^m}$, distance $\sim \dfrac{\delta_{\mathscr{C}}}{\ell\, q^{k-1}}$

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MCA of $G_{\mathsf{aff}}(\vec{v}) = \vec{v}$

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Observation 1: $((x^i)_{i\in[\ell]} \cdot \mathrm{U})|_T = (\sum_j \sum_i \phi(x^i)_j \alpha_j \mathbf{u}_i)|_T \in (\mathscr{C} \otimes \mathbb{F}_{q^k})|_T$

Observation 2: $\mathbf{u}_i \notin \mathscr{C}|_T \implies \mathbf{u}_i \notin (\mathscr{C} \otimes \mathbb{F}_{q^k})|_T$

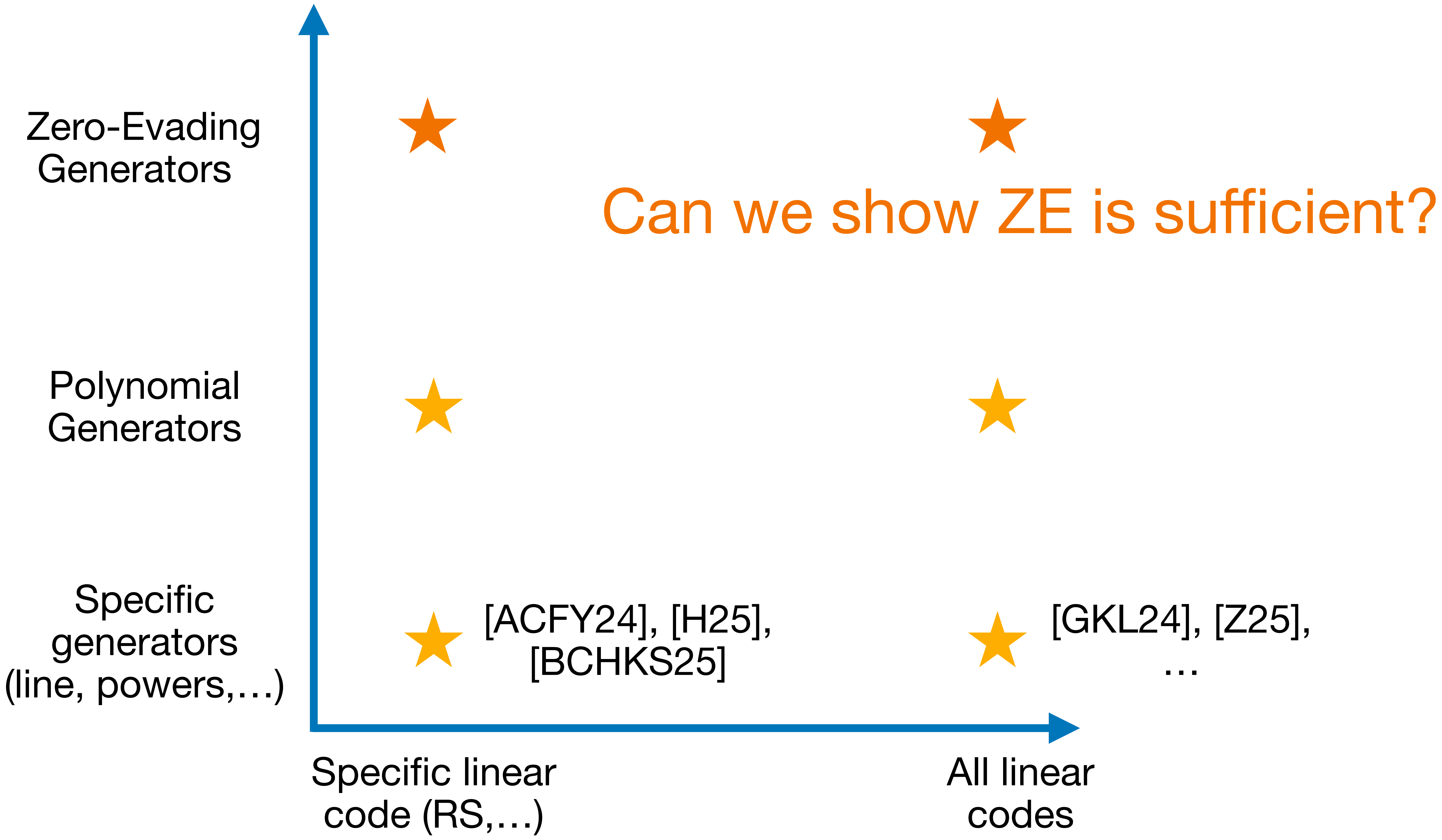
$$\begin{bmatrix} \exists T \subseteq [n], \quad |T| \geq (1-\gamma)\cdot n \\ ((x^i)_{i\in[\ell]} \cdot \mathrm{U})|_T \in (\mathscr{C} \otimes \mathbb{F}_{q^k})|_T \\ \exists i \in [\ell], \mathbf{u}_i|_T \notin (\mathscr{C} \otimes \mathbb{F}_{q^k})|_T \end{bmatrix}$$

Use univariate powers MCA over extended code

$\blacksquare$

Open Problems & Discussion

A Complete Characterization?



Is Structure Necessary?

Random generator: a uniformly random element of the set of functions $S \to \mathbb{F}^\ell$

Can show: w.p. $\geq 1 - \delta$ random generator has $\epsilon_{\mathrm{ZE}} \leq 1/|\mathbb{F}| + \mathrm{err}(\delta, |\mathbb{F}|, |S|)$

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Does a random generator have MCA with high probability?

Can We Go Beyond Johnson?

$\gamma = 0$ $\gamma = \delta_{\mathscr{C}}$

Can We Go Beyond Johnson?

Multilinear generators
and arbitrary linear codes

$\gamma = 0$

$\gamma = 1 - (1 - \delta_{\mathscr{C}})^{1/3}$

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Can We Go Beyond Johnson?

Multilinear generators and arbitrary linear codes

Polynomial generators and RS codes

Can we show something here?

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Can We Go Beyond Johnson?

The Proximity Prize

initiative by
ethereum foundation

$1,000,000

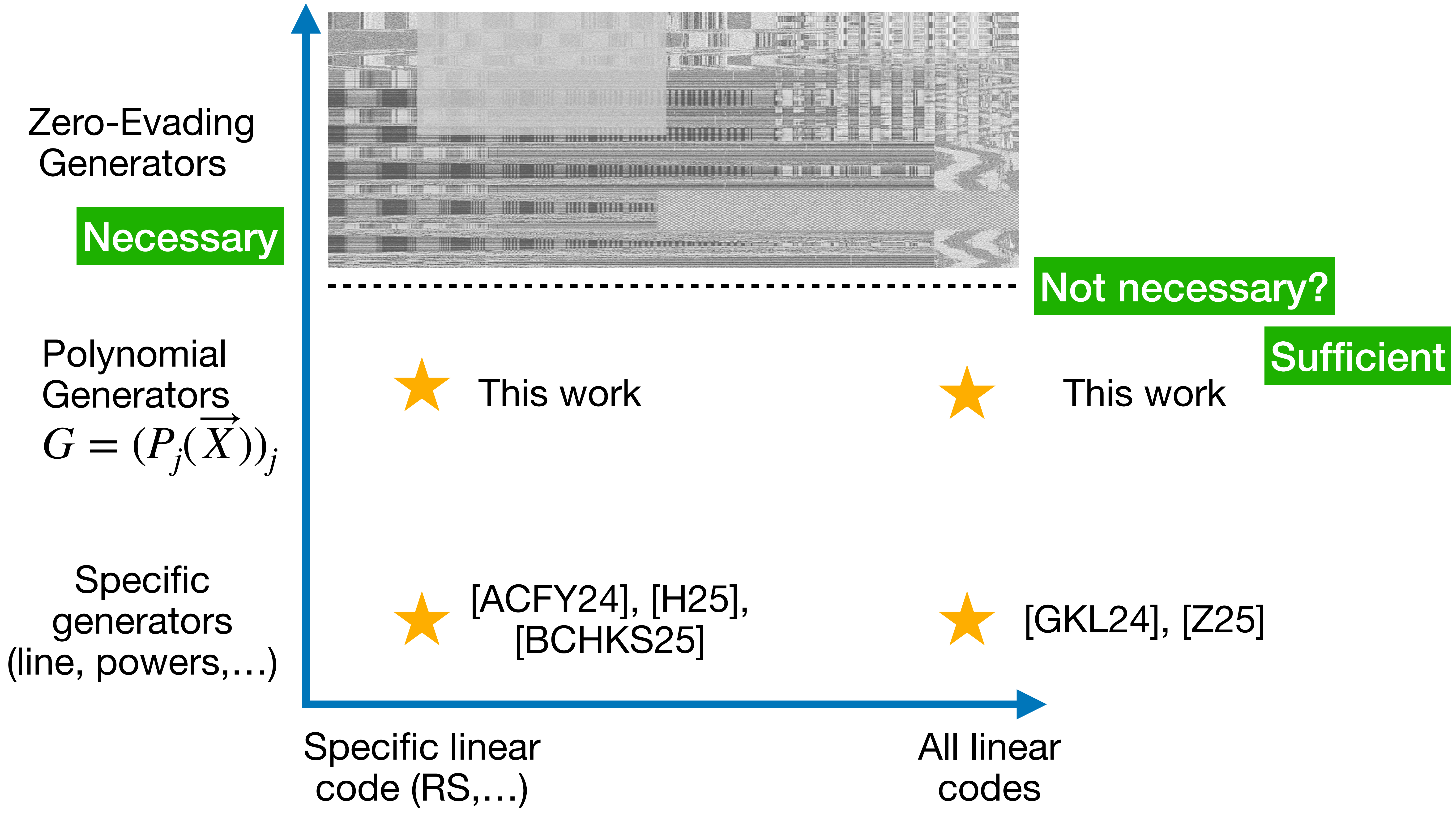
in prizes to prove (or disprove!) Reed-Solomon proximity gaps conjectures—more info soonTM

An initiative by the Ethereum Foundation to advance the foundations of modern zkVMs.

$\gamma = 0$

$\gamma = \delta_{\mathscr{C}}$

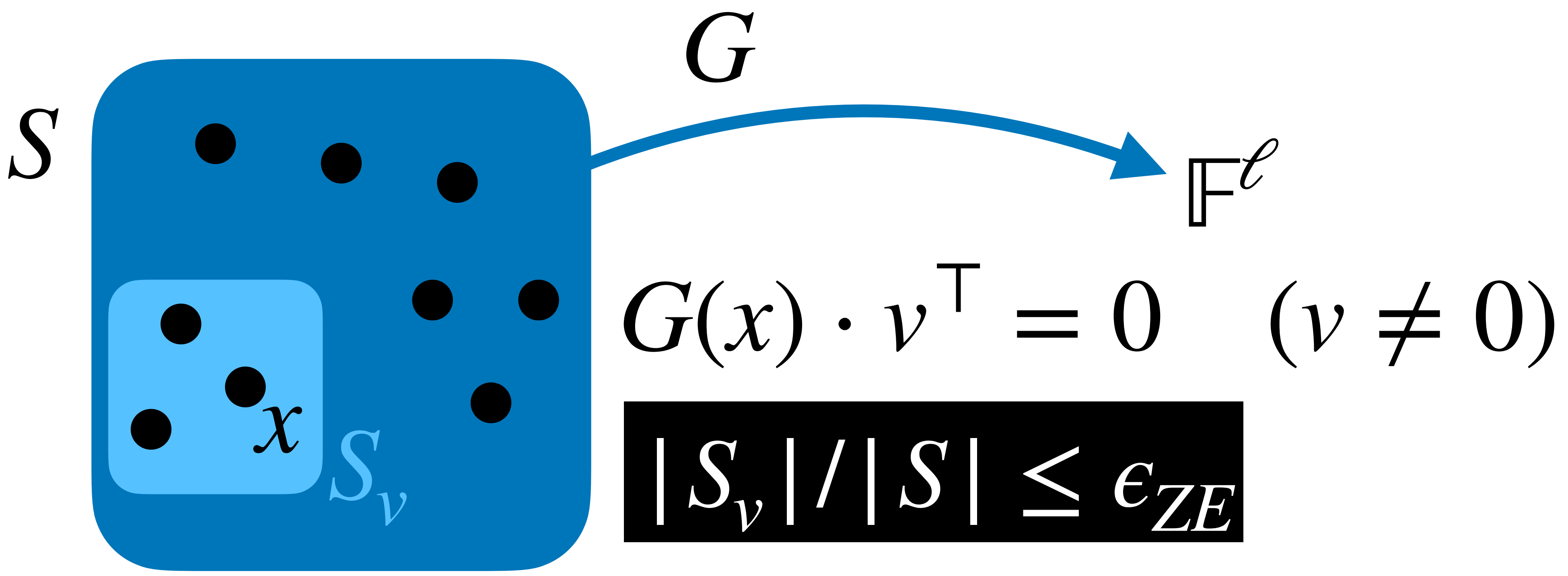
Recap



Thanks!

ZE is Necessary

Zero-evading is necessary for MCA: $\epsilon_{ZE} \leq \epsilon_{MCA}$



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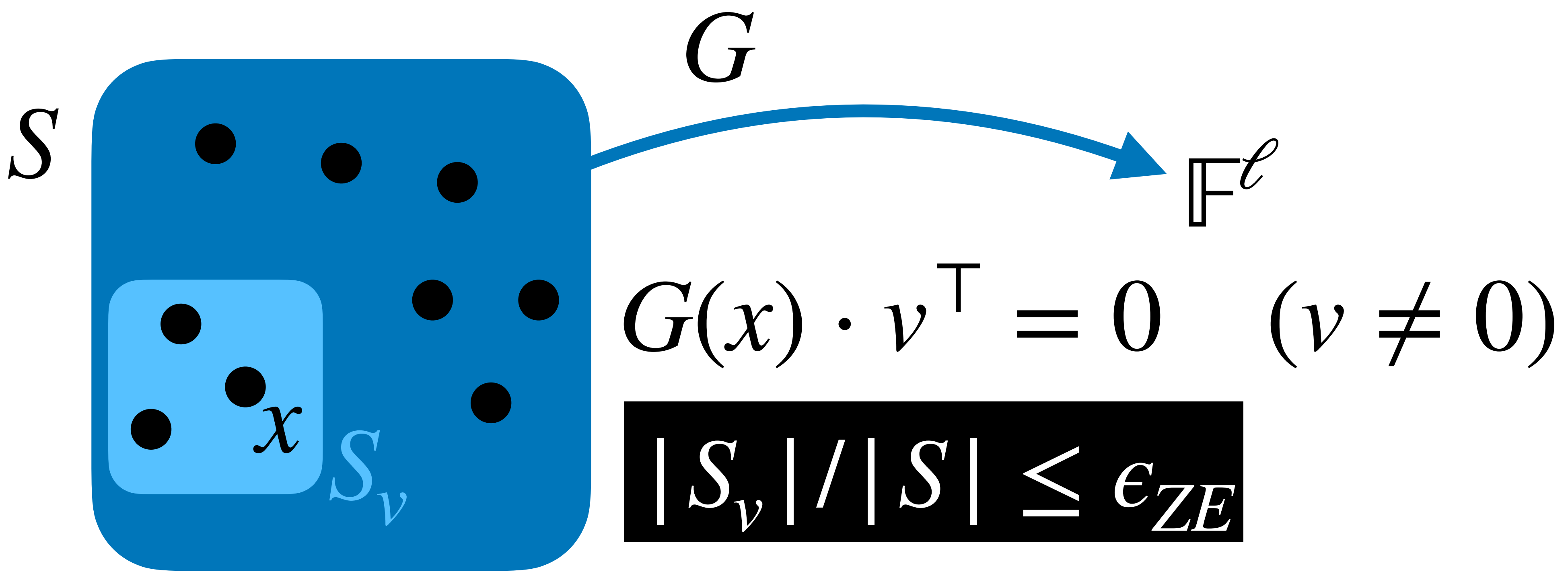

Pf sketch:

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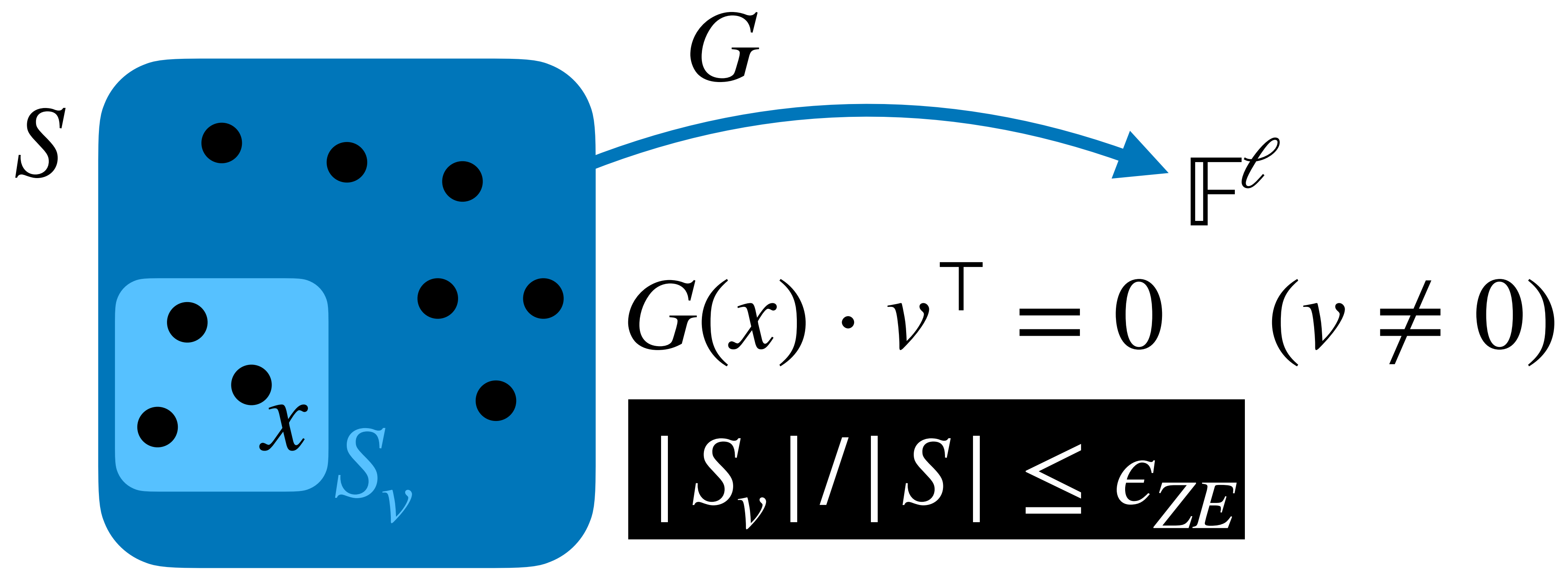

Pf sketch:

- Sufficient to show $\epsilon_{ZE} \leq \epsilon_{MCA}(0)$

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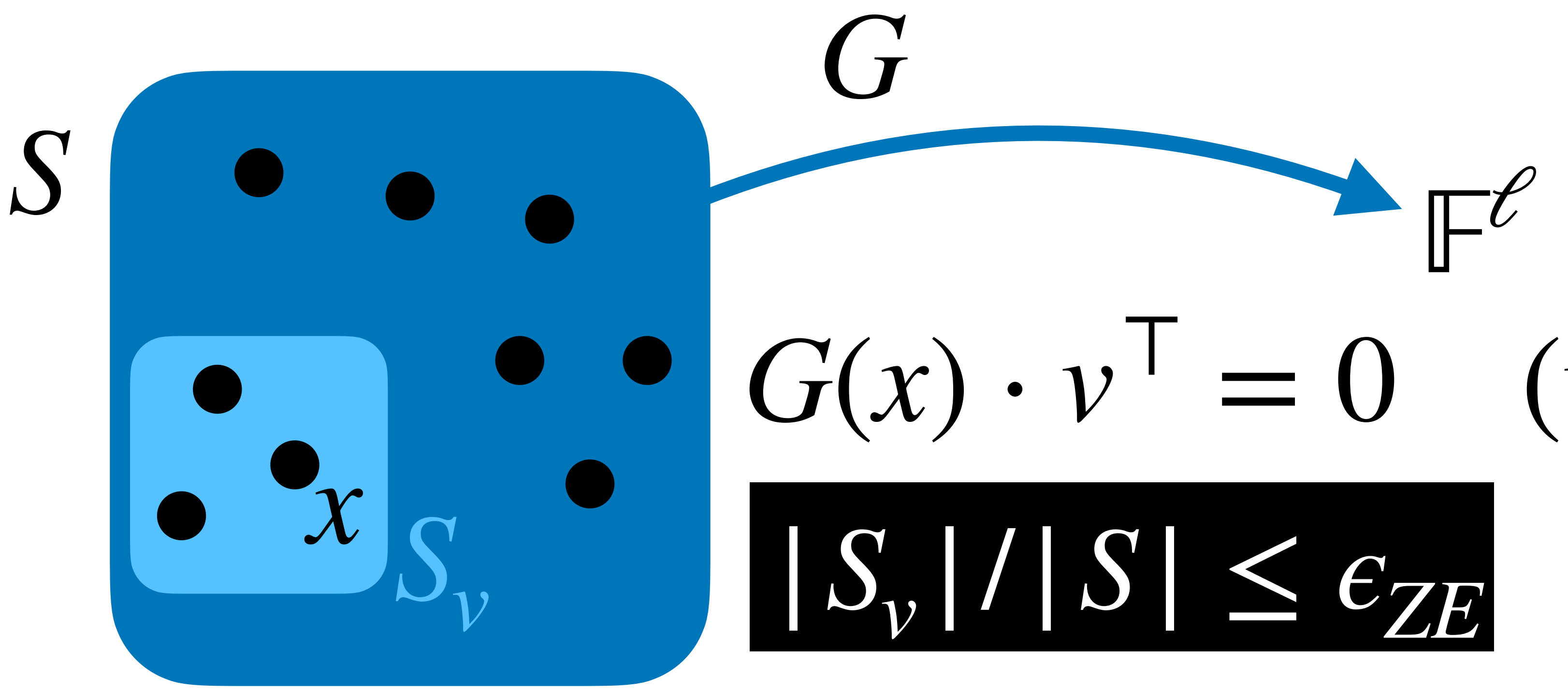

$G(x) \cdot v^\top = 0 \quad (v \neq 0)$

$|S_v|/|S| \leq \epsilon_{ZE}$

Pf sketch:

Testing membership to the code!

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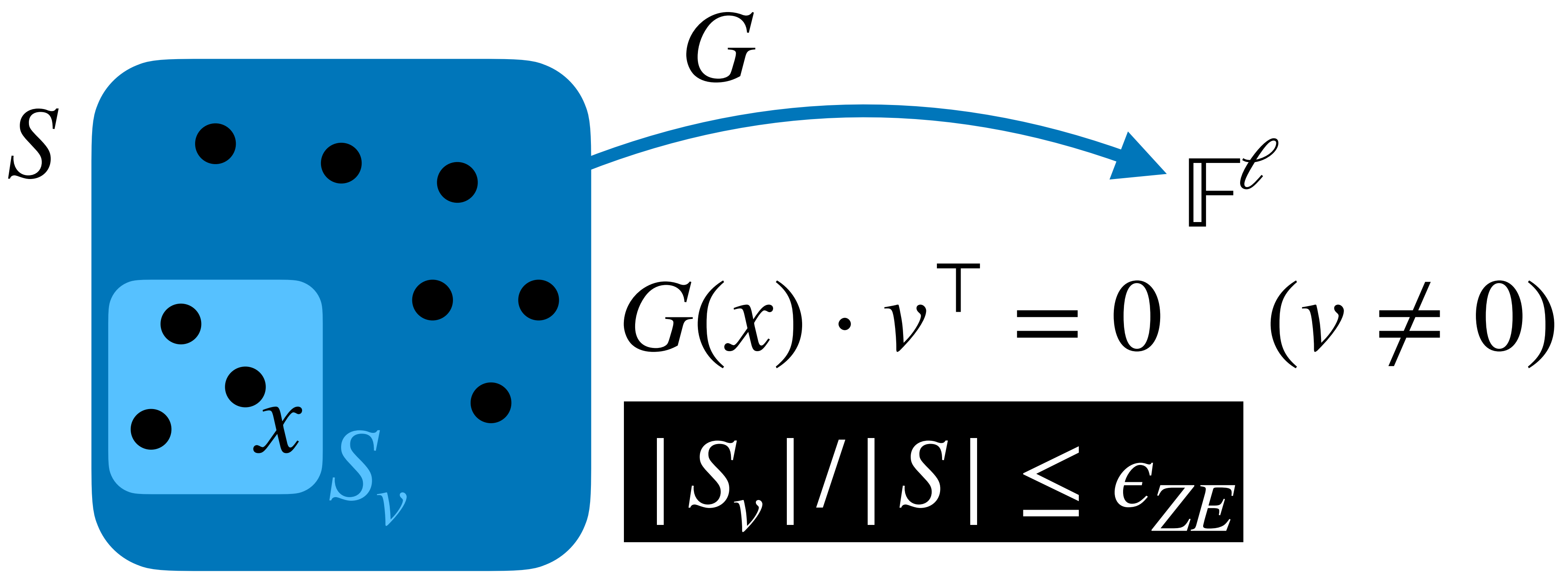

Pf sketch:

Testing membership to the code!

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- If $G(x) \cdot (u_1, \ldots, u_\ell)^{\top} \in \mathscr{C} \wedge \exists u_i \notin \mathscr{C}$, find $v \neq \mathbf{0}$ s.t. $G(x) \cdot v^{\top} = 0$

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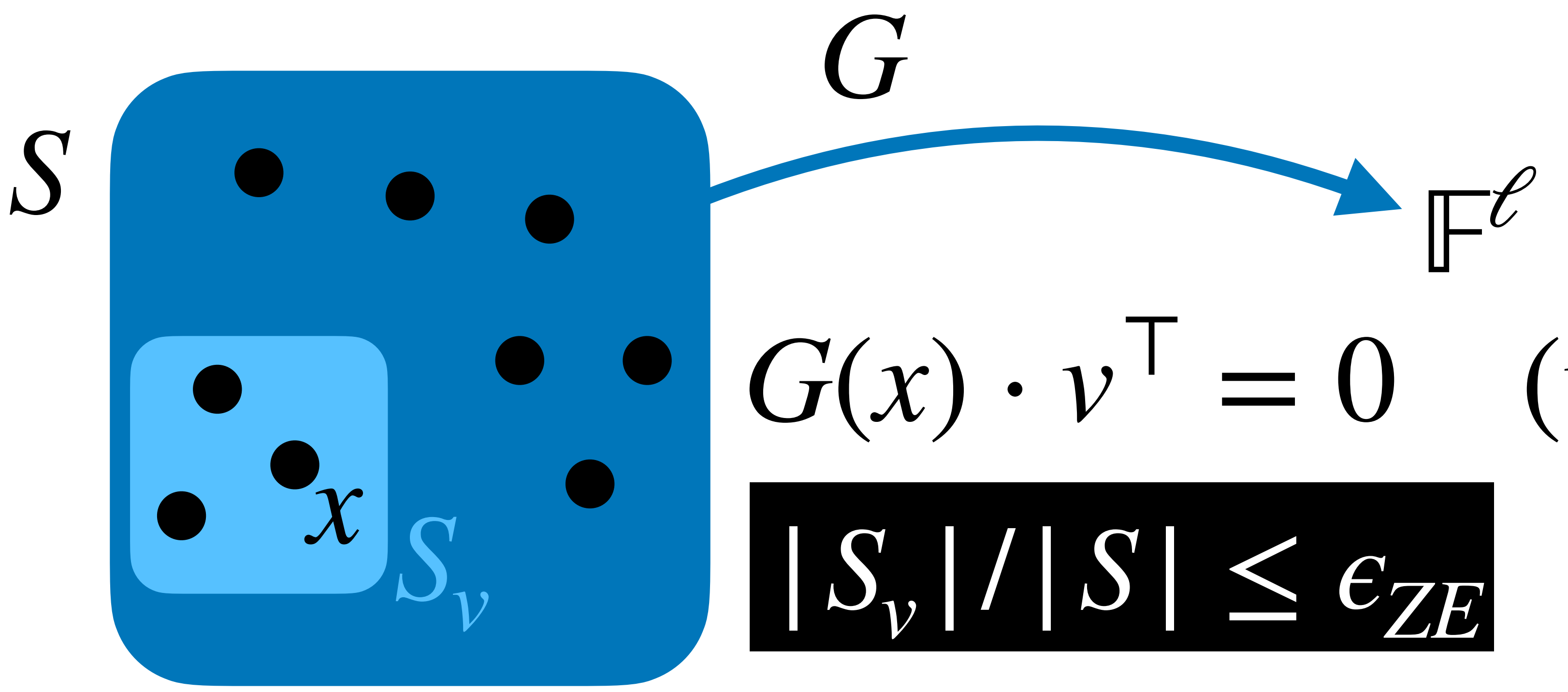

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Independent of x

- If $G(x) \cdot (u_1, \ldots, u_{\ell})^{\top} \in \mathscr{C} \wedge \exists u_i \notin \mathscr{C}$, find $v \neq \mathbf{0}$ s.t. $G(x) \cdot v^{\top} = 0$ $\blacksquare$

ZE Error Lower Bounds MCA

Lemma: $\epsilon_{ZE} = \epsilon_{MCA}(0) = \min_{\gamma \in [0,1]} \epsilon_{MCA}(\gamma)$

$$\epsilon_{MCA}(0) = \Pr_{x \in S}\begin{bmatrix} G(x) \cdot \mathbf{U} \in \mathscr{C} \\ \vee_{j \in [\ell]} \mathbf{u}_j \notin \mathscr{C} \end{bmatrix}$$

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$\vec{v} = (\mathbf{w}_1[i], \ldots, \mathbf{w}_\ell[i])$
for $i \in [r]$ s.t. $\mathbf{w}_j[i] \neq 0$

■

Zero-evading and MCA

Lemma: $\left|\{x \in S : T \subsetneq T_x\}\right| \leq (n - |T|) \cdot \epsilon_{ZE} \cdot |S|$

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Takeaway: ZE error controls agreement outside of correlated agreement set

MCA for Univariate Powers

Theorem (UDR): $G\colon S \to \mathbb{F}^{d+1}$ defined as $x \mapsto (1, x, \ldots, x^d)$ has MCA for any linear code $\mathscr{C} \subseteq \mathbb{F}^n$, with error

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$$B := \left\{ x \in S : \begin{array}{c} \exists T \subseteq [n] \\ |T| \geq n \cdot (1 - \gamma) \\ (G(x) \cdot \mathbf{U})|_T \in \mathscr{C}|_T \\ \vee_{j \in [\ell]} \mathbf{u}_j|_T \notin \mathscr{C}|_T \end{array} \right\}$$

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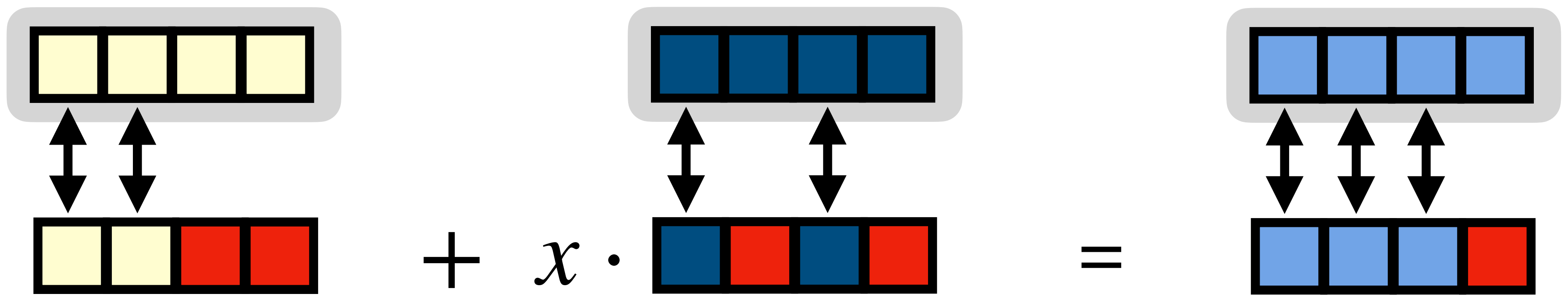

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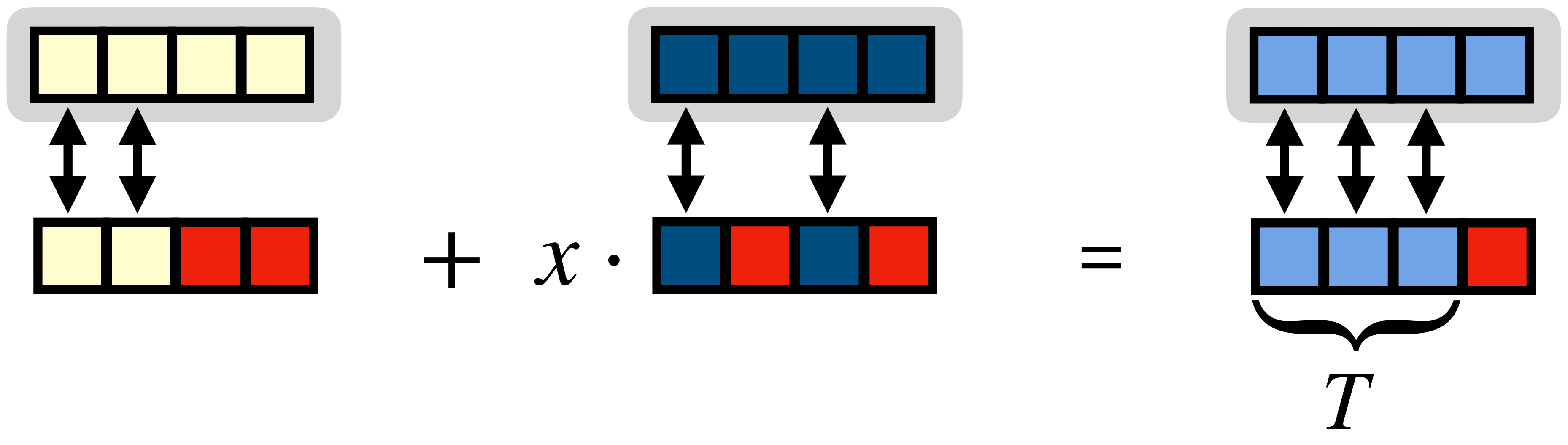

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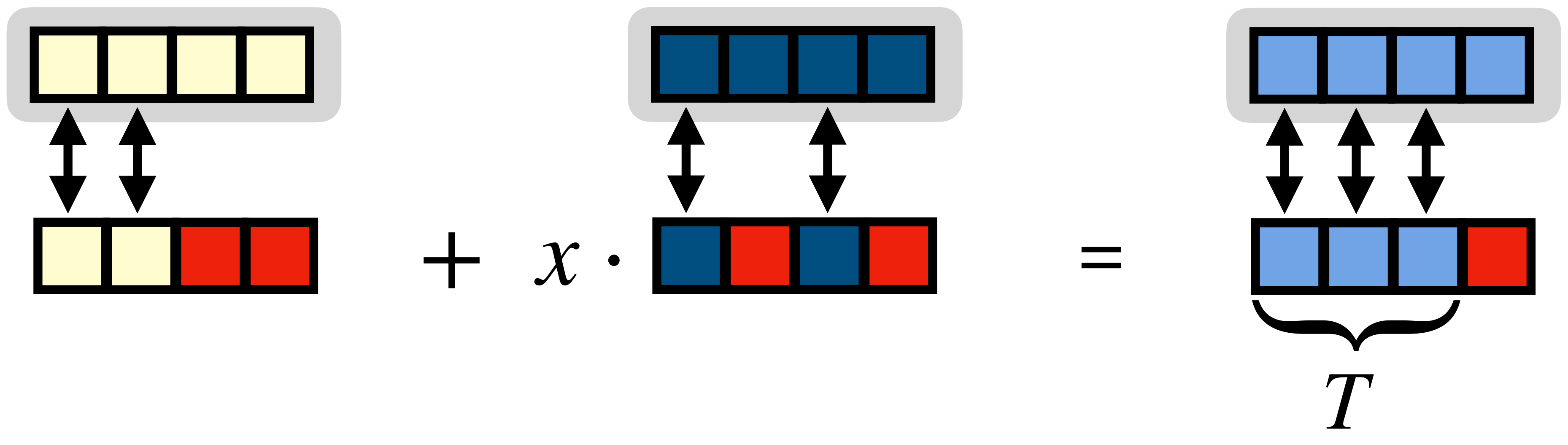

Invertibility
+triangle inequality
+UDR

Assume $|B| > n \cdot d$. We find $\mathbf{c}_1^\star, \ldots, \mathbf{c}_\ell^\star \in \mathscr{C}$:

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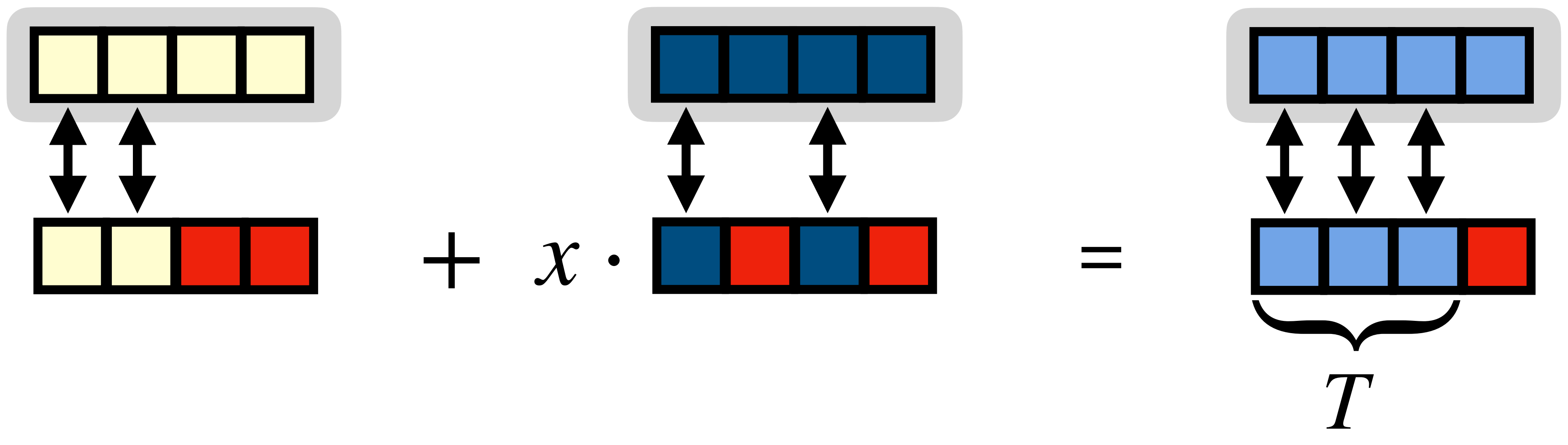

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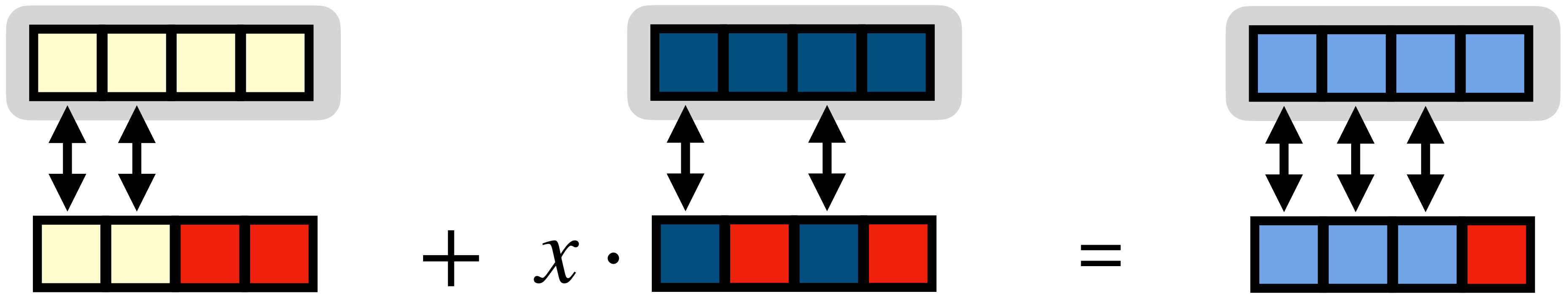

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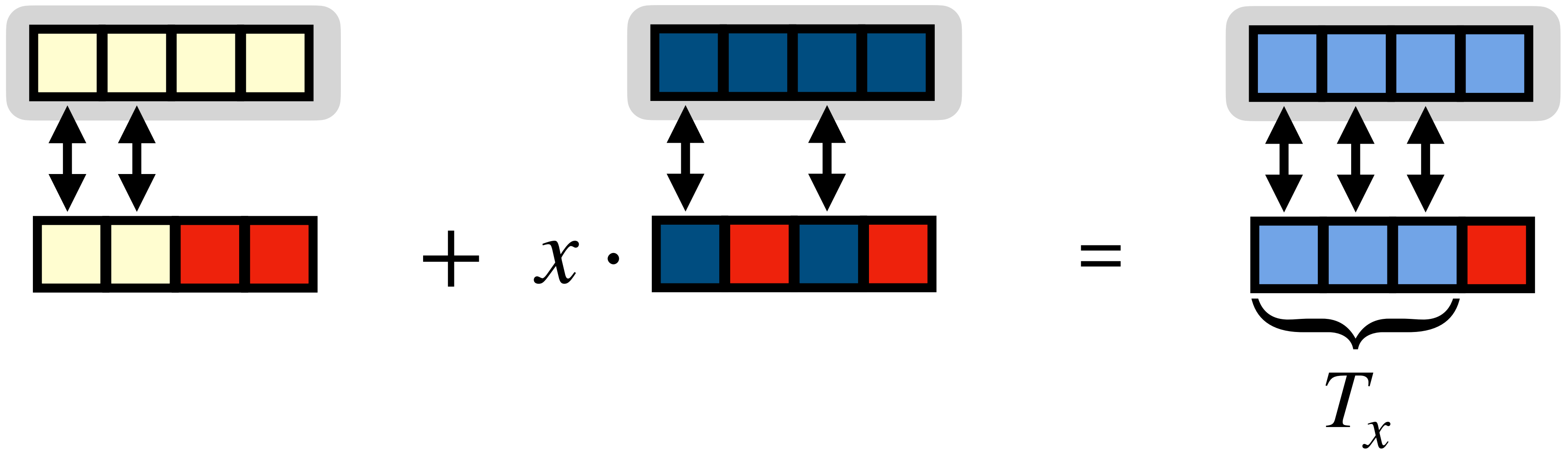

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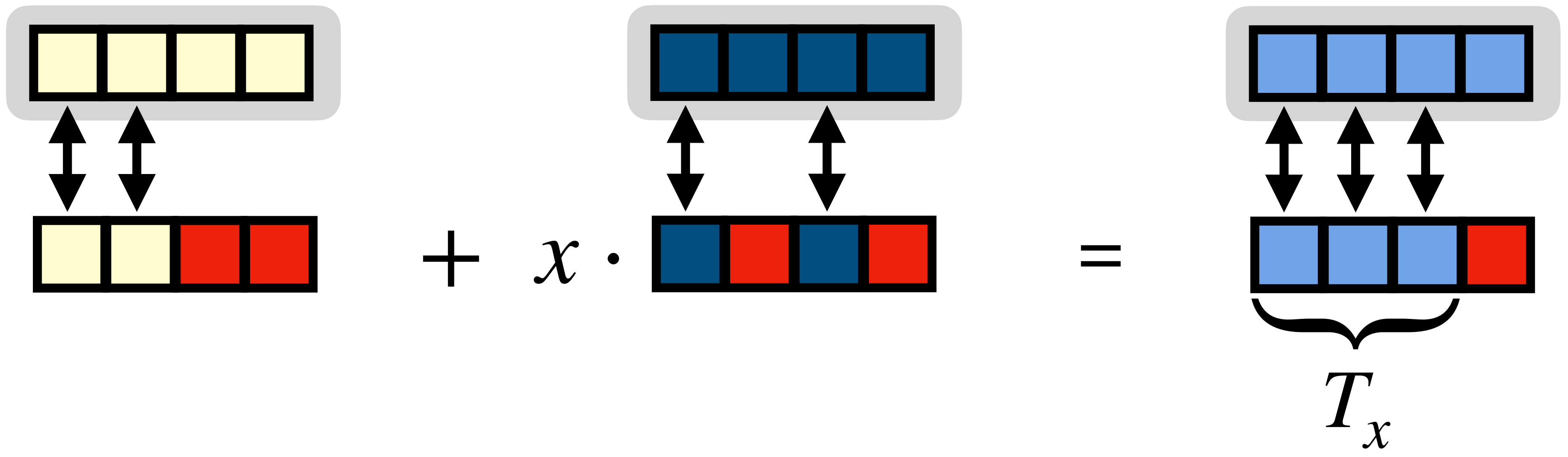
The correlated agreement set between the $\mathbf{u}_j$'s and the $\mathbf{c}_j^\star$'s



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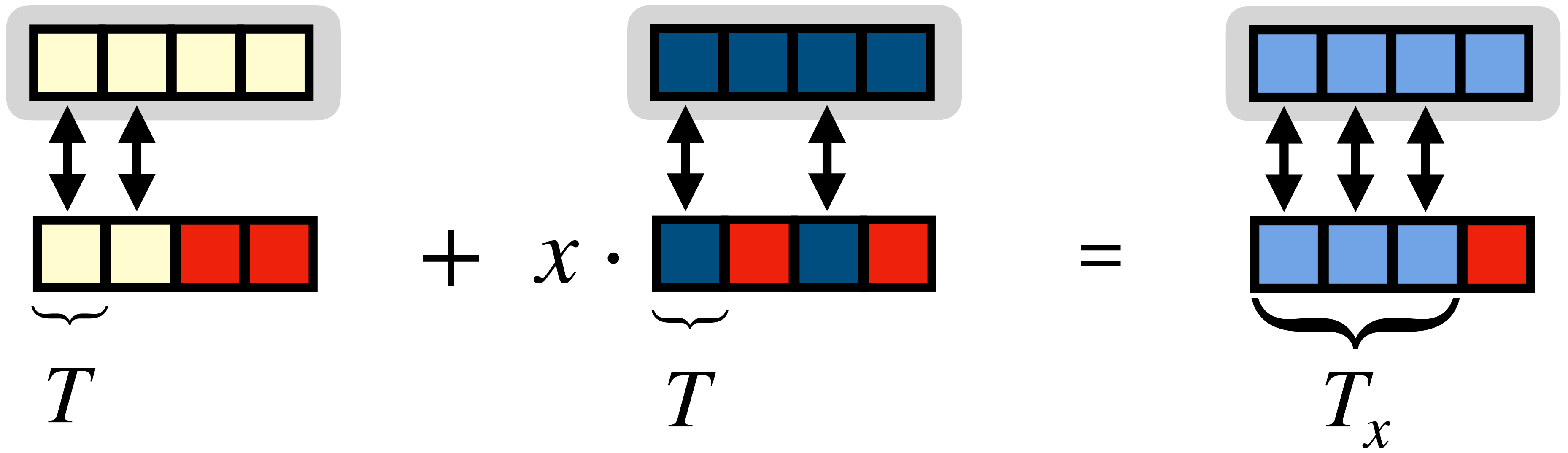

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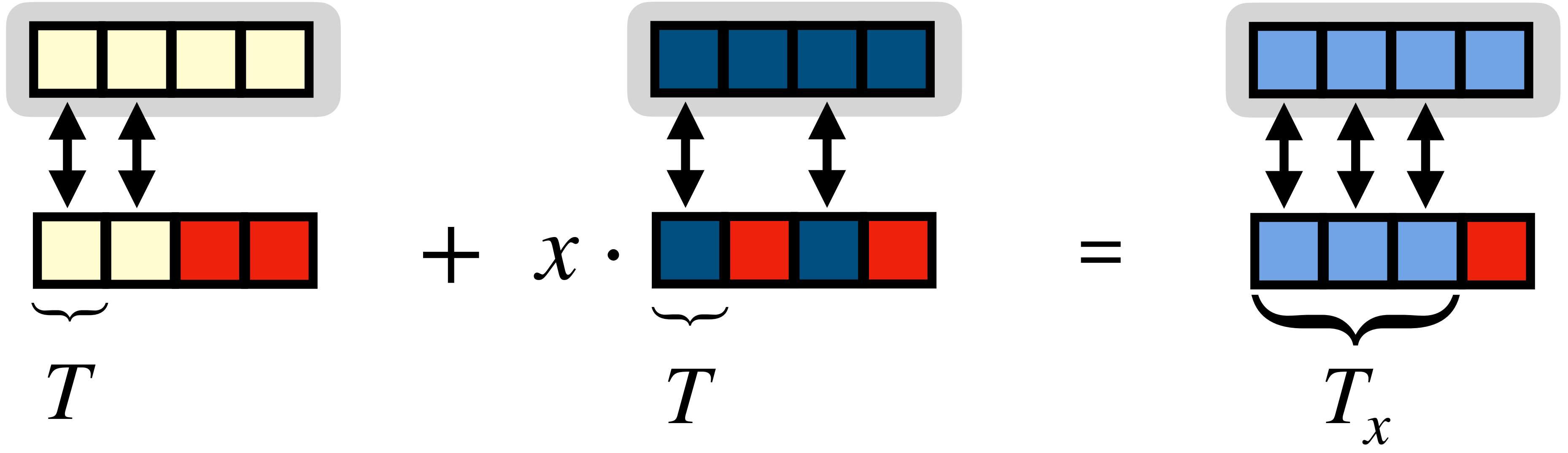

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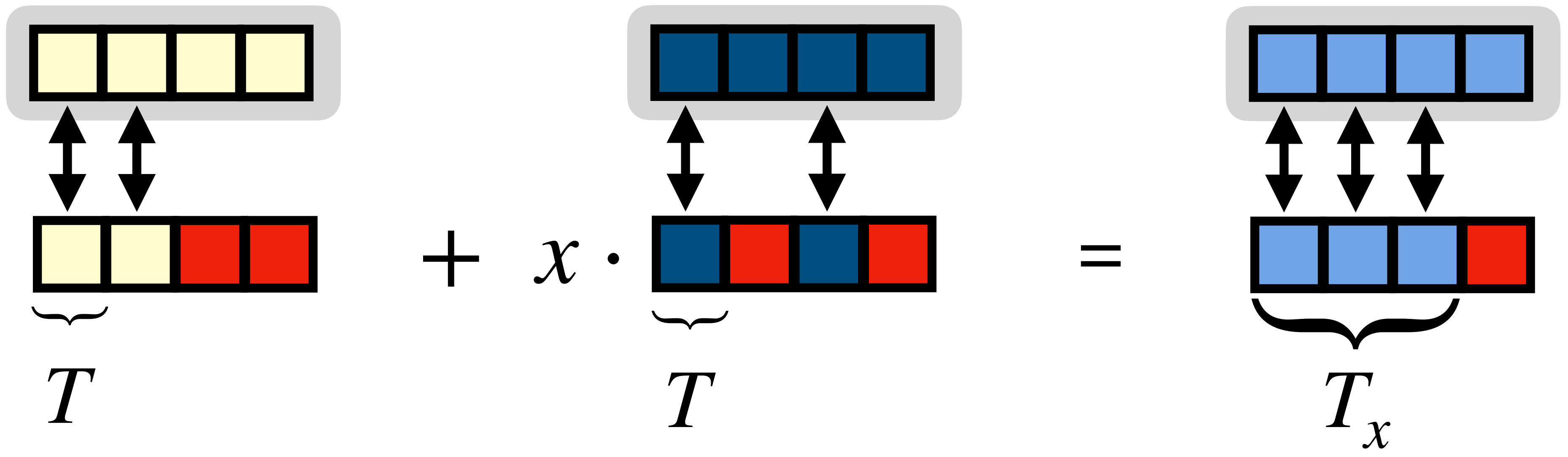

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Can show $|T| \geq n \cdot (1 - \gamma)$

Extending to list-decoding

Theorem (List-decoding): If $G: S \to \mathbb{F}^{\ell}$ is MDS (of dim. ℓ), then it has MCA for any linear code $\mathscr{C} \subseteq \mathbb{F}^n$, with error

$$\epsilon_{MCA}(\gamma) = O\left(\frac{n}{\eta}\right) \cdot \epsilon_{ZE} \ \text{ for } \ \gamma \leq 1 - (1 - \delta_{\mathscr{C}} + \eta)^{\frac{1}{\ell+1}}$$

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Counting x's such that $T \subsetneq T_x$

Beyond polynomial generators

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Lemma: this transformation (matrix-vector product) sums MCA errors