Asymptotic Rank Speedup Theorems, Revisited

CCC 2026

Josh Alman **Baitian Li**

Columbia

August 5, 2026

Asymptotic Rank, Speedup Theorems, Revisited

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- As a 3-dimensional array $T \in \mathbb{F}^n \otimes \mathbb{F}^m \otimes \mathbb{F}^p$:

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Matrix multiplication $\langle E, H, L \rangle$:

$$T(X, Y) = X \cdot Y, \quad X \in \mathbb{F}^{E \times H}, Y \in \mathbb{F}^{H \times L}.$$

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- In symmetric form (trilinear form):

$$T(\overline{x}, \overline{y}, \overline{z}) = \sum_{i,j,k} T_{ijk} x_i y_j z_k.$$

Bilinear computation vs. Turing machine

Tensors	Boolean
$T \leqslant S$	Reduction
$\langle n \rangle$	TIME$[n]$
$T \oplus S$	Solve simultaneously
$T \otimes S$	?

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$$\text{solve } S \quad \text{“}\Rightarrow\text{”} \quad \text{solve } T.$$

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- Or symmetrically:

$$T = (A \otimes B \otimes C)S$$

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- It only requires r multiplications to compute T: There are linear forms α_i, β_i $(1 \leqslant i \leqslant r)$; let

$$\gamma_i = \alpha_i(\overline{x}) \cdot \beta_i(\overline{y}),$$

every output of T is a linear combination of the γ_i.

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The rank $\mathrm{R}(T)$ of T is at most r if $T \leqslant \langle r \rangle$. Or equivalently:

- It only requires r multiplications to compute T.
- Or symmetrically, T has a *tensor rank decomposition*

$$T = \sum_{i=1}^{r} a_i \otimes b_i \otimes c_i.$$

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- Fact: $\langle n \rangle \oplus \langle m \rangle = \langle n + m \rangle$.
- Clearly, $\mathrm{R}(T \oplus S) \leqslant \mathrm{R}(T) + \mathrm{R}(S)$, which can be strict! [Shitov'19]

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- Tensor power $T^{\otimes n}$ captures "convolutions" widely used in exponential / fixed-parameter algorithms.

Asymptotic Rank

$$C(T) \geqslant \Omega(\mathrm{R}(T))$$

- Rank counts #multiplications and thus lower bounds circuit complexity.

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$$\mathrm{R}(T^{\otimes n}) \leqslant \mathrm{R}(T)^n$$

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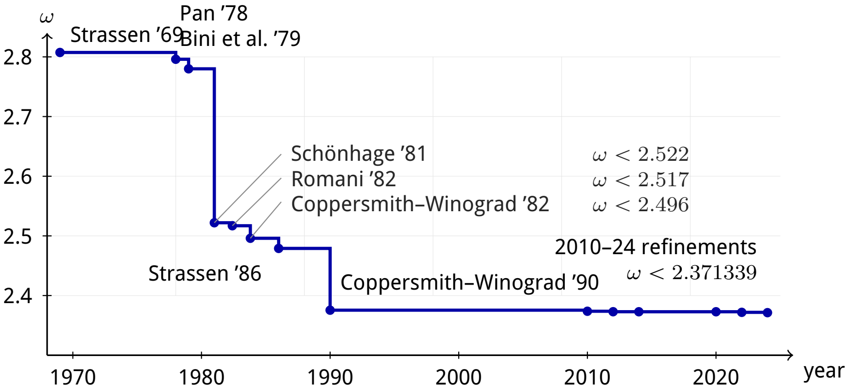
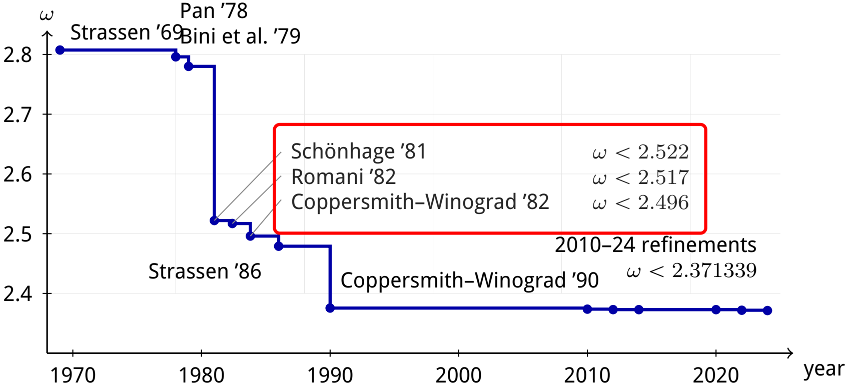
$$\underbrace{\underset{\sim}{\mathrm{R}}(T)}_{\textbf{asymptotic rank}} \coloneqq \lim_{n\to\infty} \mathrm{R}(T^{\otimes n})^{1/n} \leqslant \mathrm{R}(T^{\otimes n})^{1/n}$$

Matrix multiplication exponent over time



ω is the number satisfying $\underset{\sim}{\mathrm{R}}(\langle n, n, n\rangle) = n^{\omega}$.

Matrix multiplication exponent over time



We will focus on the story behind this part.

Methods for matrix multiplication

In 1981, Schönhage proved the following beautiful theorem:

Theorem (Asymptotic Sum Inequality [Schönhage'81])

If we can bound a direct sum

$$\bigoplus_{i=1}^{\ell} \langle E_i, H_i, L_i \rangle \trianglelefteq \langle r \rangle,$$

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- The celebrated *Schönhage direct sum* (1981)

$$\langle n, 1, m \rangle \oplus \langle 1, (n-1)(m-1), 1 \rangle \trianglelefteq \langle nm + 1 \rangle$$

gives $\omega < 2.55$ by instantiating $n = m = 4$.

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- Backbone of all later developments, including the *Laser Method*.

A legacy issue

In his 1981 paper, Schönhage used a "*rather formidable identity*":

$$\langle 1,5,20\rangle \oplus \langle 10,2,5\rangle \oplus \langle 10,10,1\rangle \trianglelefteq \langle 132\rangle \oplus 6\langle 1,1,2\rangle. \qquad (*)$$

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  which gives $\omega < 2.516094.$
  - They both claimed a proof independently, but for some reason, neither was ever published.

An analogy with physics

The asymptotic sum inequality

$$\underset{\sim}{\mathrm{R}}\left(\bigoplus_i \langle n_i, n_i, n_i\rangle\right) \leqslant r$$

$$\Updownarrow$$

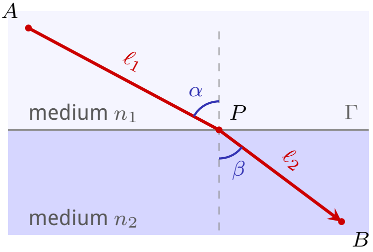
$$\sum_i n_i^\omega \leqslant r$$

$$\Updownarrow$$

$$\sum_i \underset{\sim}{\mathrm{R}}(\langle n_i, n_i, n_i\rangle) \leqslant r$$

"The best way to do repeated tasks is to do them independently."

Fermat's principle of optics



"Light is minimizing its travel time."

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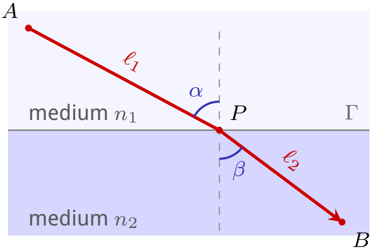

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It shouldn't be merely a verifiable fact.

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Higher principle

The **asymptotic spectrum duality**

[Strassen'88]

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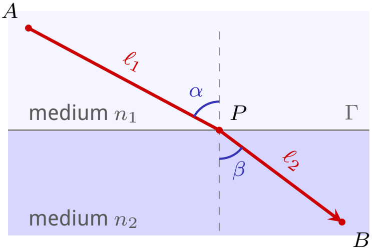

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Higher principle

The **principle of least action**

Asymptotic spectrum duality

The asymptotic rank $\underset{\sim}{\mathrm{R}}$ is defined by *minimization*:

$$\underset{\sim}{\mathrm{R}}(T) = \lim_{n \to \infty} \mathrm{R}(T^{\otimes n})^{1/n} = \inf_n \mathrm{R}(T^{\otimes n})^{1/n}.$$

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Strassen developed the dual theory of *asymptotic spectrum*:

Theorem (Strassen, 1988)

The asymptotic rank $\underset{\sim}{\mathrm{R}}(T)$ can be expressed as a ***maximization****:*

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- $\phi(S) \leqslant \phi(T)$ *if* $S \leqslant T$ *(respecting order)*
- $\phi(S \oplus T) = \phi(S) + \phi(T)$
- $\phi(S \otimes T) = \phi(S) \cdot \phi(T)$ *(respecting algebra)*

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The asymptotic rank $\underset{\sim}{\mathrm{R}}(T)$ can be expressed as a ***maximization****:*

$$\underset{\sim}{\mathrm{R}}(T) = \max_{\phi\in\mathcal{X}} \phi(T),$$

where $\mathcal{X}$ is the set of nonzero ϕ: $\{\text{tensors}\} \to \mathbb{R}_{\geq 0}$ satisfying:

- $\phi(S) \leq \phi(T)$ if $S \leq T$ *(respecting order)*
- $\phi(S \oplus T) = \phi(S) + \phi(T)$
- $\phi(S \otimes T) = \phi(S) \cdot \phi(T)$ *(respecting algebra)*

$\mathcal{X}$: the asymptotic spectrum of tensors

Asymptotic spectrum duality cont.

Strassen duality

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  - **This work**: derive new asymptotic rank upper bounds.

Strassen calculus

A toy problem

If a tensor T has restrictions

$$T \leqslant \langle r \rangle \oplus \langle s, 1, 1 \rangle, \quad T \leqslant \langle r \rangle \oplus \langle 1, s, 1 \rangle, \quad T \leqslant \langle r \rangle \oplus \langle 1, 1, s \rangle,$$

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ϕ is additive: $\phi(S \oplus T) = \phi(S) + \phi(T)$.

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Theorem (This work)

For any q, we have $\underset{\sim}{\mathrm{R}}(\mathrm{cw}_q) < q + 2$. In particular, $\underset{\sim}{\mathrm{R}}(\mathrm{cw}_2) < 3.931$.

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Another technical ingredient: **Speedup theorems**.

Speedup theorems

Theorem ([Coppersmith–Winograd'82] [Strassen'88])

If a degeneration $T \trianglelefteq S$ satisfies certain technical conditions, we can "speed up" the identity to

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where $t > s$.

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  1. Generalize previous speedup theorems to be more modular
  2. Complement with Strassen calculus, which helps obtain a *quantitative version* and a concrete bound on cw_2.

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Thanks