Constant-Depth Circuits for Polynomial GCD over Any Characteristic

Understanding the complexity of symmetric polynomials

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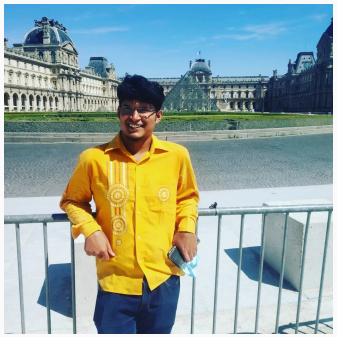
Joint work with



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Polynomial GCD

$$f(x) = \prod_{i \in [d]} (x - \alpha_i) = x^d + f_{d-1} x^{d-1} + \ldots f_0 \in \mathbb{F}[x]$$

$$g(x) = \prod_{i \in [d]} (x - \beta_i) = x^d + g_{d-1} x^{d-1} + \ldots g_0 \in \mathbb{F}[x]$$

$\gcd(f, g)$ is the highest degree polynomial that divides both f and g.

Algorithmic question: given coefficients of f and g, compute the coefficients of $\gcd(f, g)$.

Polynomial GCD

Euclid

?

Parallel = circuits

Boolean functions: shallow Boolean circuits.

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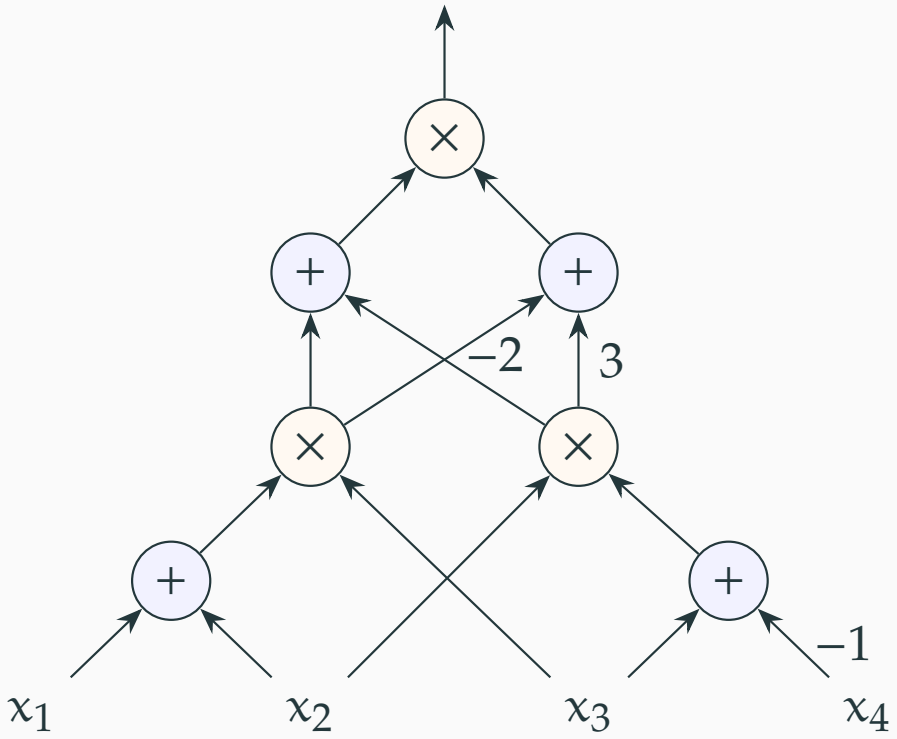
Boolean functions: shallow Boolean circuits.

Polynomials: shallow arithmetic circuits.

$$x_1^2x_3^2 + 3x_1x_3^2x_2 - x_1x_3x_2x_4 - 4x_2^2x_3^2 + 11x_2^2x_3x_4 - 6x_2^2x_4^2$$



Technicality

$\gcd$ is not continuous.

Example.

$$\gcd(x-\alpha, x-\beta) = \begin{cases} 1, & \text{if } \alpha \neq \beta, \\ x-\alpha, & \text{if } \alpha = \beta. \end{cases}$$

To avoid this issue, we will focus on the Resultant.

Resultant

$$f(x) = (x - \alpha_1)(x - \alpha_2)\dots(x - \alpha_d)$$

$$g(x) = (x - \beta_1)(x - \beta_2)\dots(x - \beta_d)$$

Resultant

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$$\mathrm{Res}(f, g) := \prod_{i \in [d], j \in [d]} (\alpha_i - \beta_j)$$

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Given the coefficients of f and g, how easy is the Resultant?

Resultant is a polynomial

$$\mathrm{Res}(f, g) := \prod_{i \in [d], j \in [d]} (\alpha_i - \beta_j)$$

Fact: $\mathrm{Res}(f, g) \in \mathbb{F}[f_0, \ldots, f_{d-1}, g_0, \ldots, g_{d-1}]$

$$\mathrm{Res}(f, g) = \det(\mathrm{Syl}(f, g))$$

Complexity of the Resultant

$\mathrm{Res}(f, g)$: size $\mathrm{poly}(d)$, depth $O(\log^2 d)$ [Csanky 1976, Berkowitz 1983, . . .].

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Theorem (Andrews and Wigderson 2024)

Over fields with char$(\mathbb{F})$ *zero or large,* $\mathrm{Res}(f, g)$ *has size* $\mathrm{poly}(d)$*, depth* $O(1)$ *circuits.*

Also for $\gcd$ and many other problems.

Our results

Theorem (BKRRSS)

Over any large enough field, $\mathrm{Res}(f, g)$ *has size* $\mathrm{poly}(d)$*, depth* $O(1)$ *circuits.*

Also for $\gcd$.

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Symmetric polynomials

$f(z_1, \ldots, z_n)$ is a symmetric polynomial if for any permutation $\pi : [n] \to [n]$,

$$f(z_{\pi(1)}, \ldots, z_{\pi(n)}) = f(z_1, \ldots, z_n).$$

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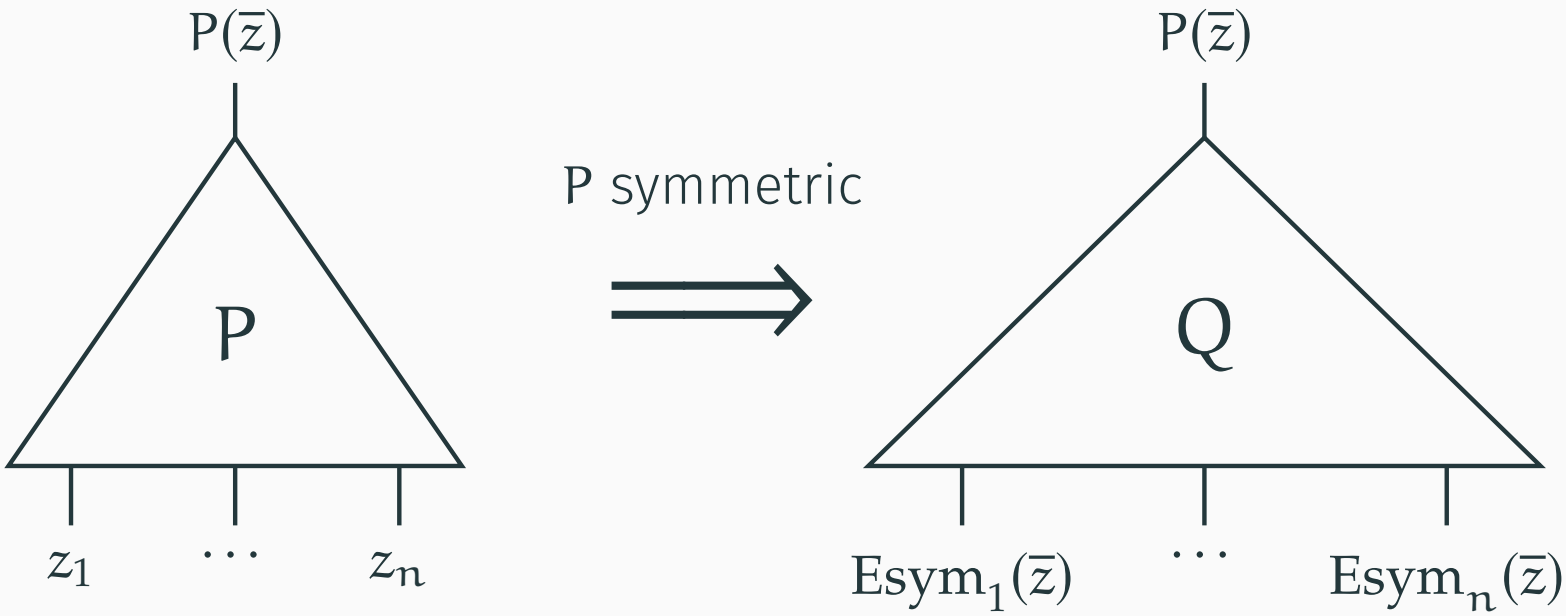
Example:

$$\mathrm{Esym}_d(\bar{z}) = \sum_{\substack{S \subseteq [n] \\ |S| = d}} \prod_{j \in S} z_j$$

Fundamental Theorem of Symmetric Polynomials



Q unique.

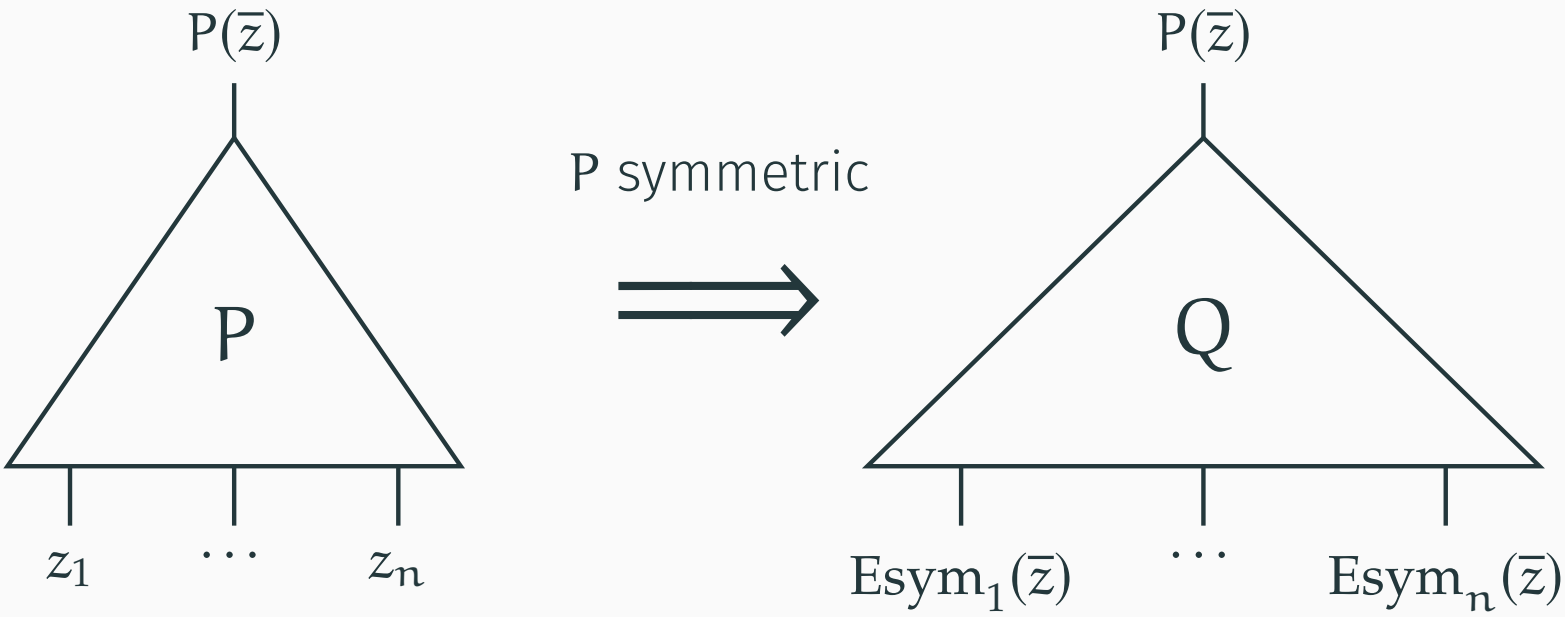
Vieta's theorem

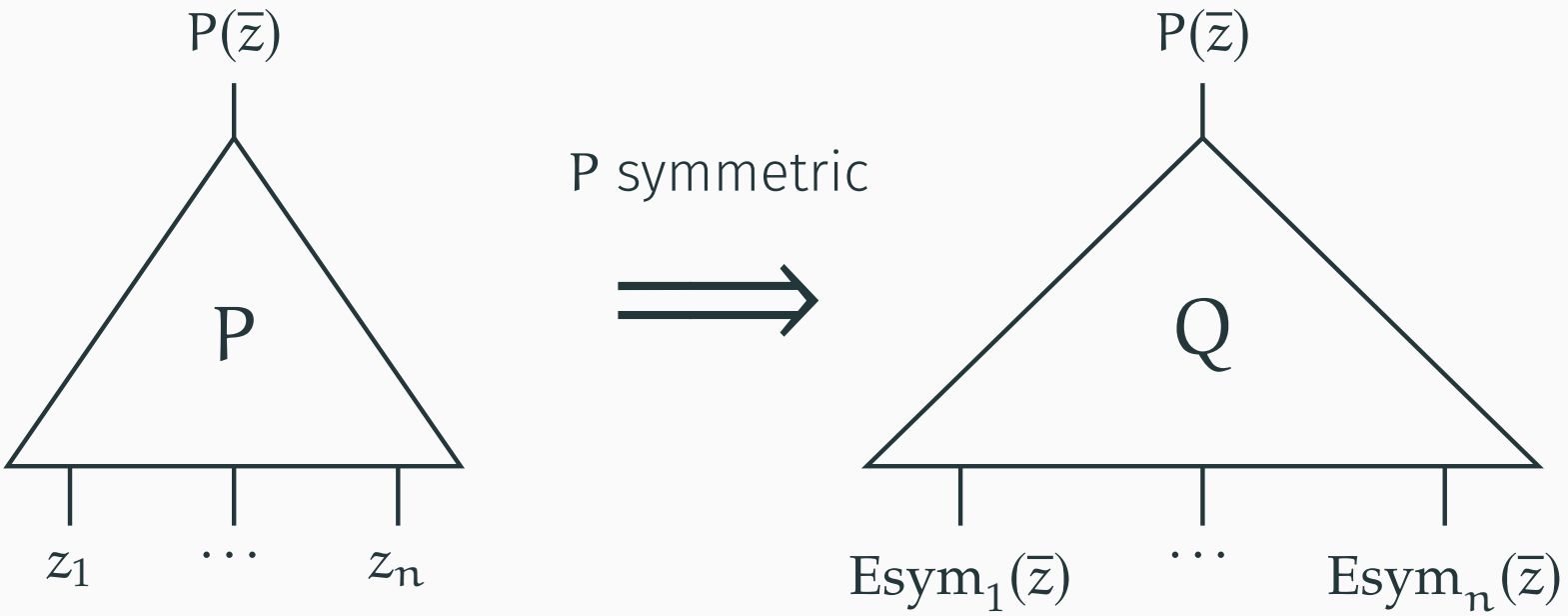
If

$$f(x) = \prod_{i \in [d]} (x - \alpha_i) = x^d + f_{d-1} x^{d-1} + \dots f_0$$

then for $i \in \{0, 1, \dots, d-1\}$,

$$f_i = (-1)^{d-i} \mathrm{Esym}_{d-i}(\alpha_1, \dots, \alpha_d)$$

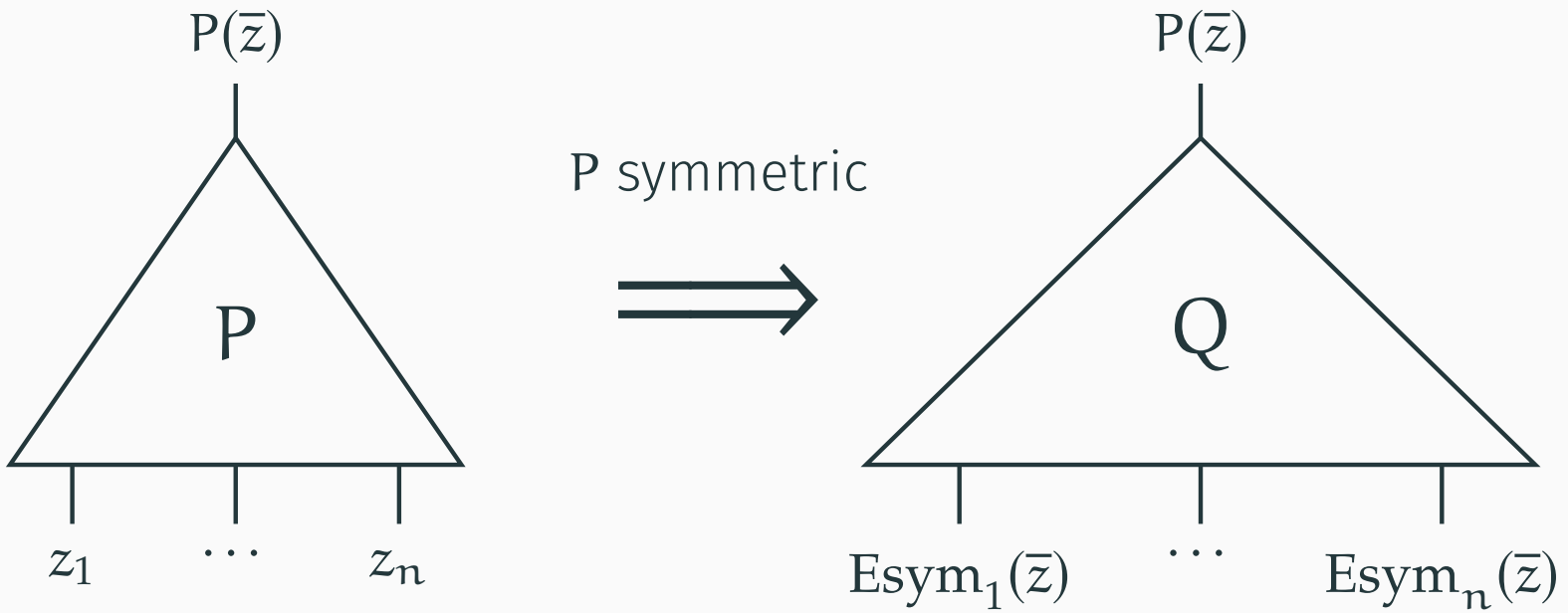

P(z̄)
P
z1
· · ·
zn
P symmetric
P(z̄)
Q
Esym1(z̄)
· · ·
Esymn(z̄)



size s, degree d

size $\mathrm{poly}(s, d, n)$

Our result — BKRRSS



size s, degree d, depth Δ

size $\mathrm{poly}(s, d, n)$, depth $\Delta + O(1)$

Over large enough fields. Also for multi-symmetric polynomials.

Resultant in constant-depth

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Proof.

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Symmetric with respect to $(\alpha_1, \ldots, \alpha_d)$ and $(\beta_1, \ldots, \beta_d)$.

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$$\mathrm{Res}(f, g) = Q(\mathrm{Esym}_1(\overline{\alpha}), \ldots, \mathrm{Esym}_d(\overline{\alpha}), \mathrm{Esym}_1(\overline{\beta}), \ldots, \mathrm{Esym}_d(\overline{\beta}))$$

where Q has size $\mathrm{poly}(d)$, depth $O(1)$.

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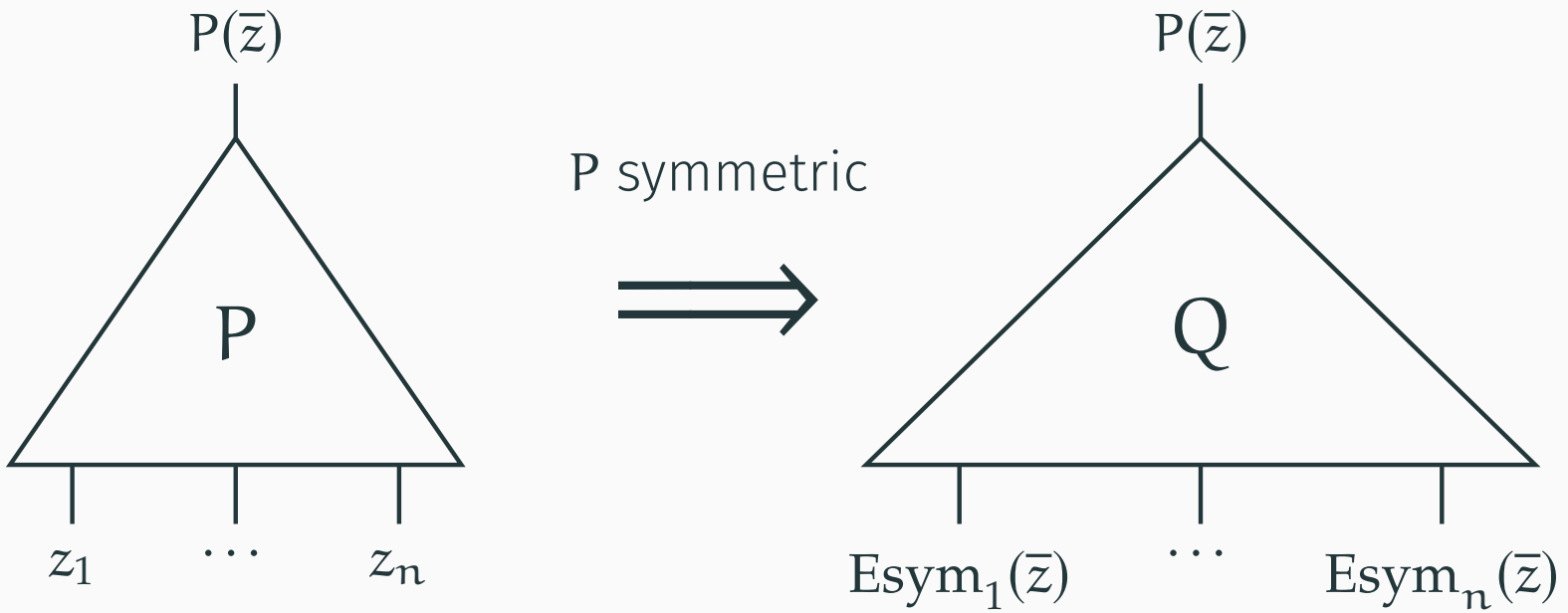
Apply Vieta.

□

Our result — BKRRSS

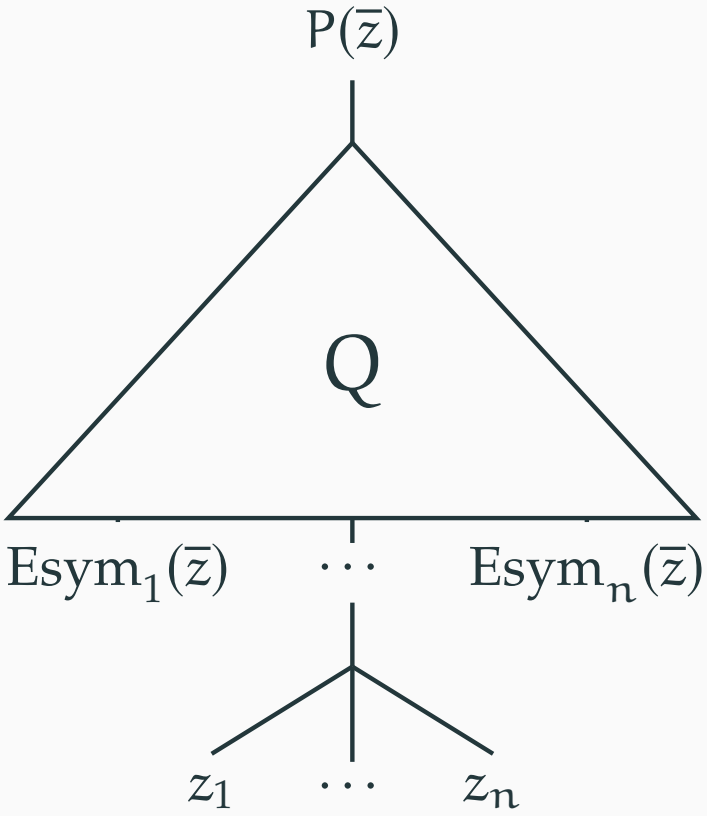


size s, degree d, depth Δ

size $\mathrm{poly}(s, d, n)$, depth $\Delta + O(1)$

Over large enough fields. Also for multi-symmetric polynomials.

Proof idea


$P(\bar{z})$
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$\cdots$
$\mathrm{Esym}_n(\bar{z})$
z_1
$\cdots$
z_n

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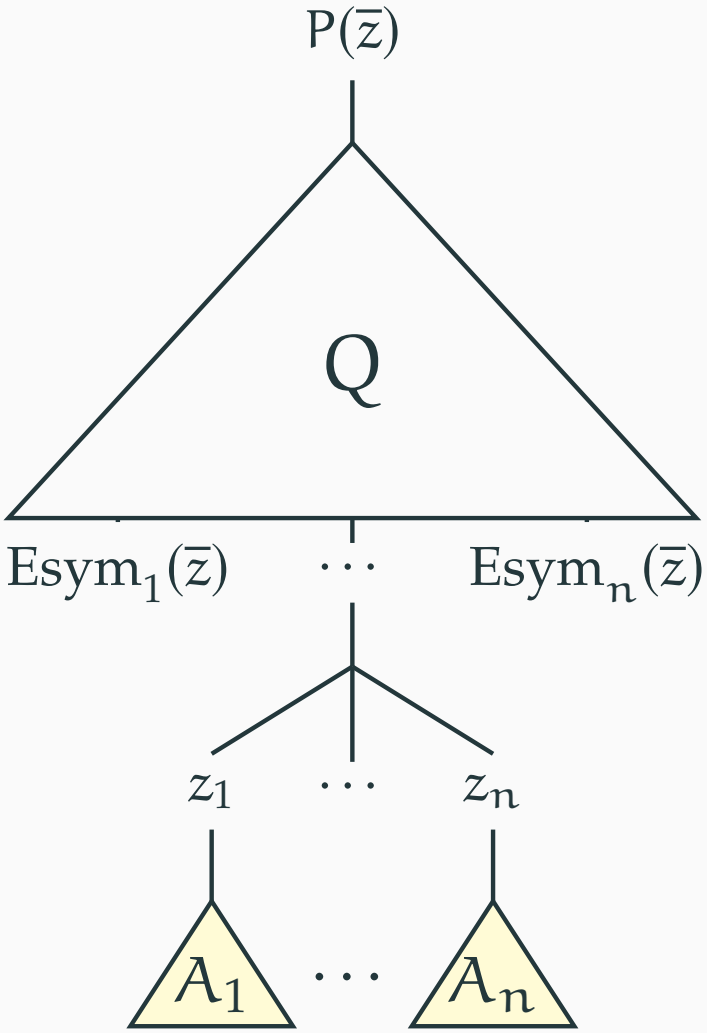
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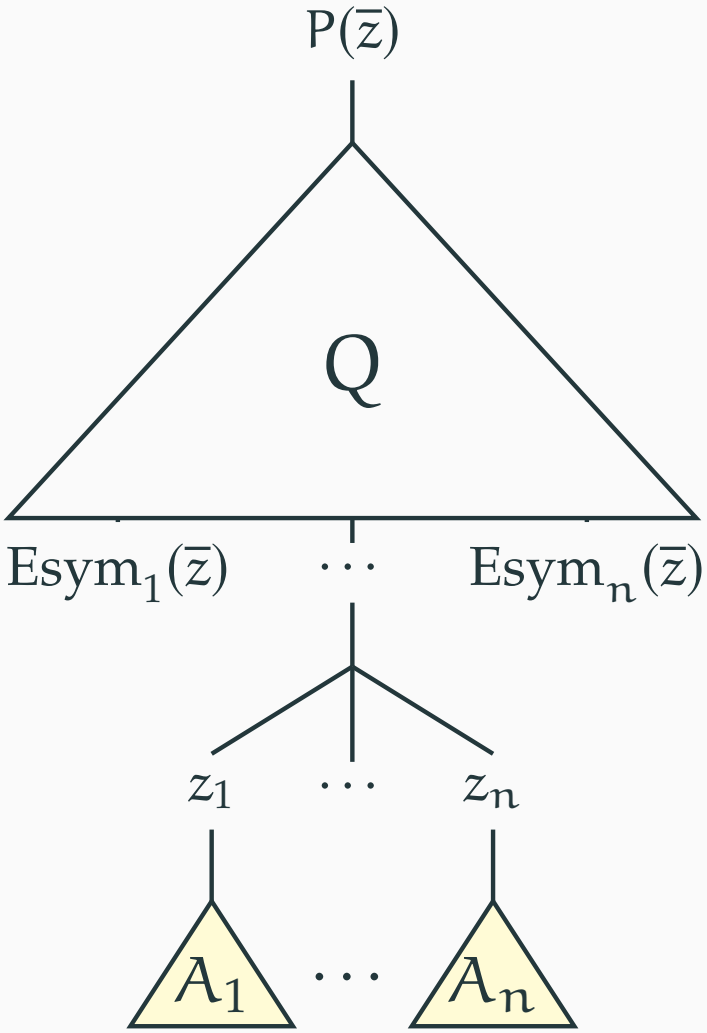
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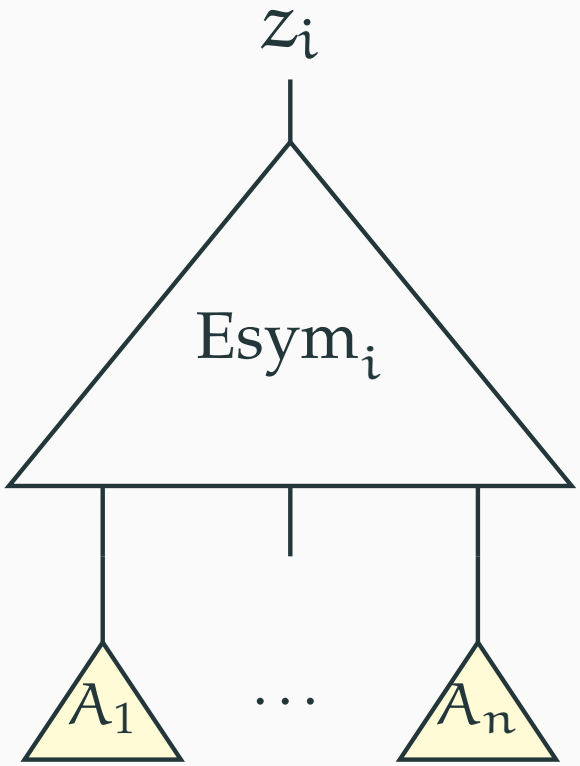
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By Vieta, same as computing the roots of

$$R(\bar{z}, x) = x^n + z_1 x^{n-1} + z_2 x^{n-2} + \cdots + z_n$$

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Recent structural results for roots/factors of multivariate polynomials [BKRRSS25]: we can express these roots $A_i(\bar{z})$ using circuits of size $\mathrm{poly}(n)$ and depth $O(1)$.

□

Remarks

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Can extend to GCD using ideas similar to Andrews-Wigderson.

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Works for other models like arithmetic formulas, arithmetic branching programs, etc.

Takeaway

Efficient symmetric expression using roots

$$\Longrightarrow$$

efficient expression using coefficients.

Open questions

Half-GCD algorithm: sequential near-linear time algorithm.
Constant-depth circuit of near-linear size for polynomial GCD?

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