Rank bounds and polynomial-time PIT for $\Sigma^k\Pi\Sigma\Pi^2$ circuits

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CCC
August, 2026

Joint with

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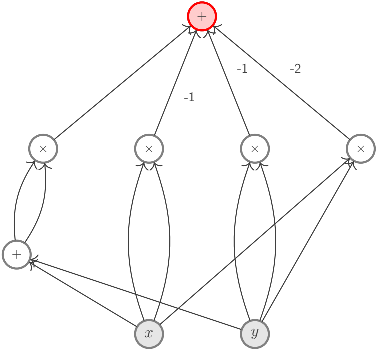
Algebraic Circuits

Natural model for computing polynomials.

$$(x+y)^2 - x^2 - y^2 - 2xy$$



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- By depth reduction **[Agrawal Vinay 2008, Koiran 2012, Tavenas 2015]**: "morally" sufficient to study depth-4 circuits

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    $$x^2 - y^2 - (x-y)(x+y) \equiv 0 \qquad \text{(identity)}$$

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$\Longrightarrow$: SG circuit $\iff$ $\forall i, j,\ \exists k$ s.t. $c_k \in \mathrm{span}_{\mathbb{C}}\{a_i, b_j\}$

Sylvester-Gallai type theorems

- **[Sylvester 1893, Melchior 1940, Erdös 1943, Gallai 1944]** Suppose $\mathcal{F} \subseteq \mathbb{R}^2$ is a finite set such that for every $p, q \in \mathcal{F}$ there is a third point $r \in \mathcal{F}$ on the line joining p, q.

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- **[Hansen 1966]** (Higher dimension): $\mathcal{F} \subseteq \mathbb{R}^N$ finite, such that for every k points $p_1, \ldots, p_k \in \mathcal{F}$ and any other point $p \in \mathcal{F}$, either $p \in \text{aff-span}_{\mathbb{R}}(p_1, \ldots p_k)$ or $|\text{aff-span}_{\mathbb{R}}(p_1, \ldots p_k, p) \cap \mathcal{F}| > k + 1$.

  Then, $\dim \text{span}_{\mathbb{R}}(\mathcal{F}) \leq SG(k, \mathbb{R})$.

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- What about depth-4 circuits?

Rank of depth-4 circuits

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The ideal need not be prime. We cannot "split" the product.

Radical EK configurations

- **Definition** (Radical EK): Finite, pairwise disjoint sets $A, B, C \subseteq \mathbb{C}[x_1, \ldots, x_n]$ of non-associate irreducible forms of degree $\leq d$ form a d-EK configuration if for all $a \in A, b \in B$
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  - Needs approximate quadratic Hansen's theorem.

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Simple minimal $\Sigma^r\Pi\Sigma$ identity $\Phi = \sum_{i=1}^{r} T_i$

Pairwise rank: $r_{i,j}(\Phi) := \dim \mathrm{span}_{\mathbb{C}} \{T_i \Delta T_j\}$

Some pairwise rank is small

Set forms in $T_i \Delta T_j$ to "generic" scalars.
Merge the two gates

Large pairwise ranks, and $\exists$ "rank preserving" linear form

$\exists \ell \in T_i$, s.t. setting $\ell = 0$ preserves large pairwise rank.
Reduce top fanin

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Linear forms satisfy Hansen's incidence conditions.
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In the first two cases, apply induction on the top fanin; in the last case, Hansen's theorem directly gives low rank.

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  Then, there exists a constant-dimensional space V such that every $Q \in \mathcal{F}$ is close to V.

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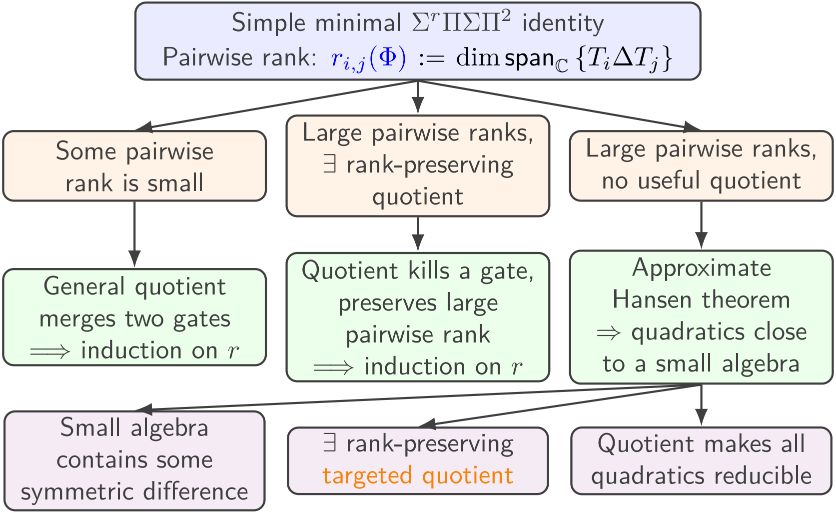

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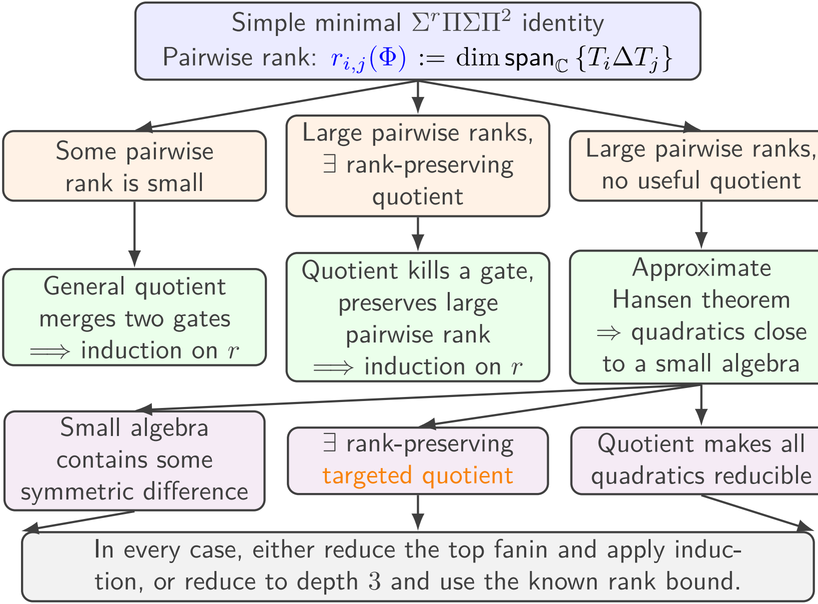

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Thank You

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