Polynomial Identity Testing for Read-4 Arithmetic Formulas

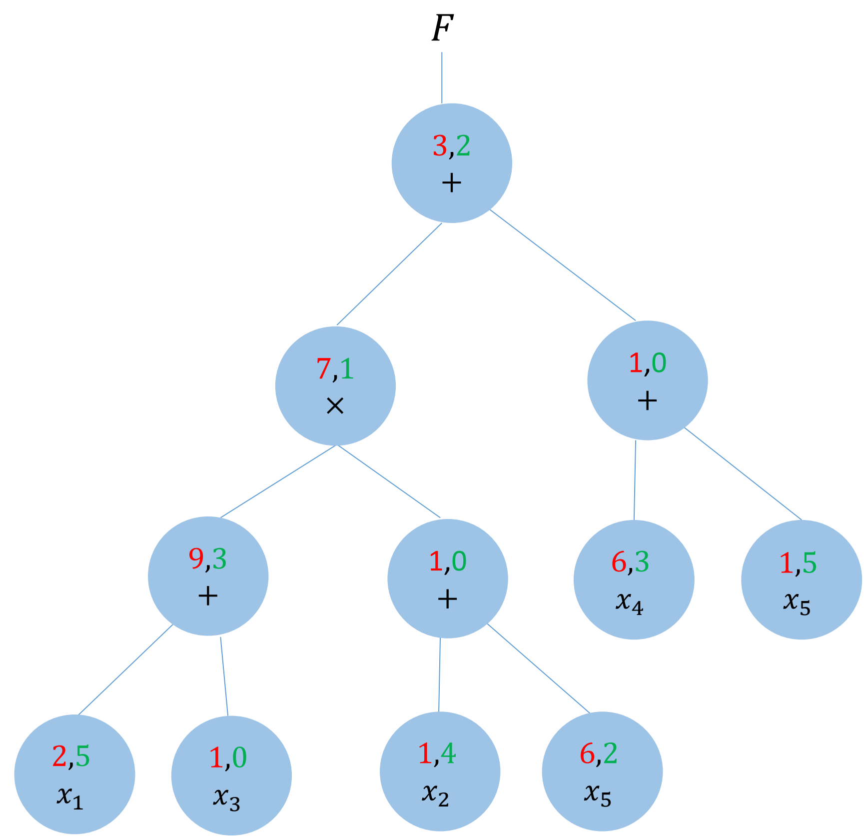
Nimrod Kaplan Amir Shpilka

Overview

- Background
- Our results
- Why is Read-4 interesting?
- Some new techniques
- Future directions – higher reads

Bounded Read Arithmetic Formulas

- Binary tree
- Inputs are variables $x_1, x_2, \ldots x_n = \vec{x}$
- Gates labeld with $+$ and $\times$
- Multiplicative and additive constants
- Computation in the "natural" way
- Read-k formula: Every variable $x_i \in \boldsymbol{x}$ appears at most k times (RkF, ROF)



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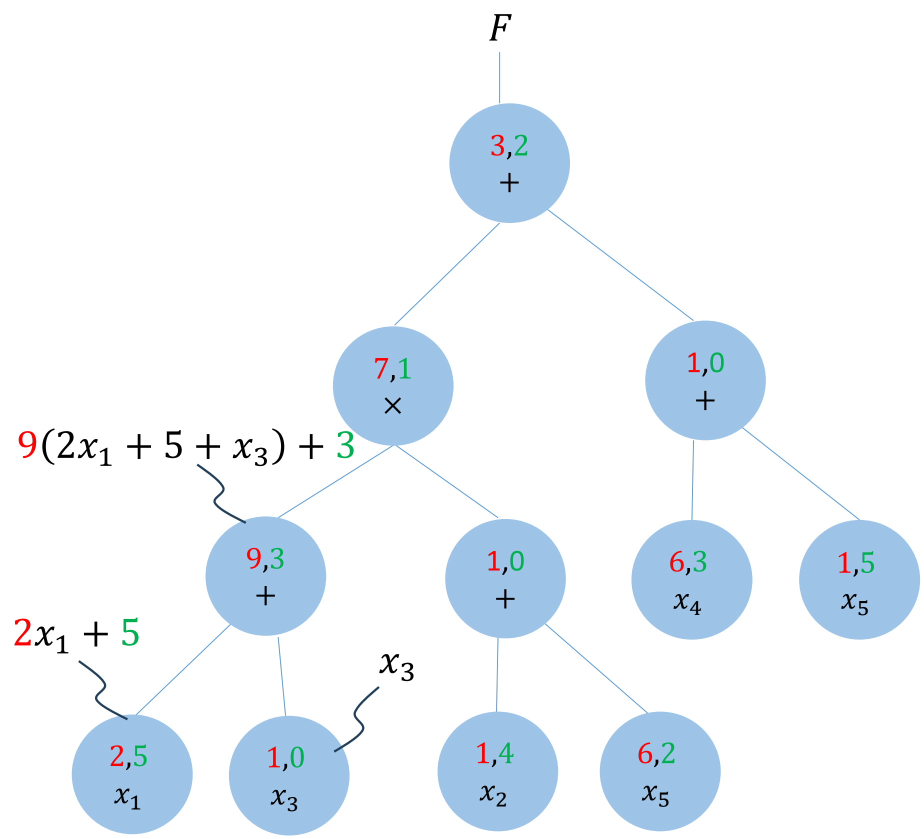

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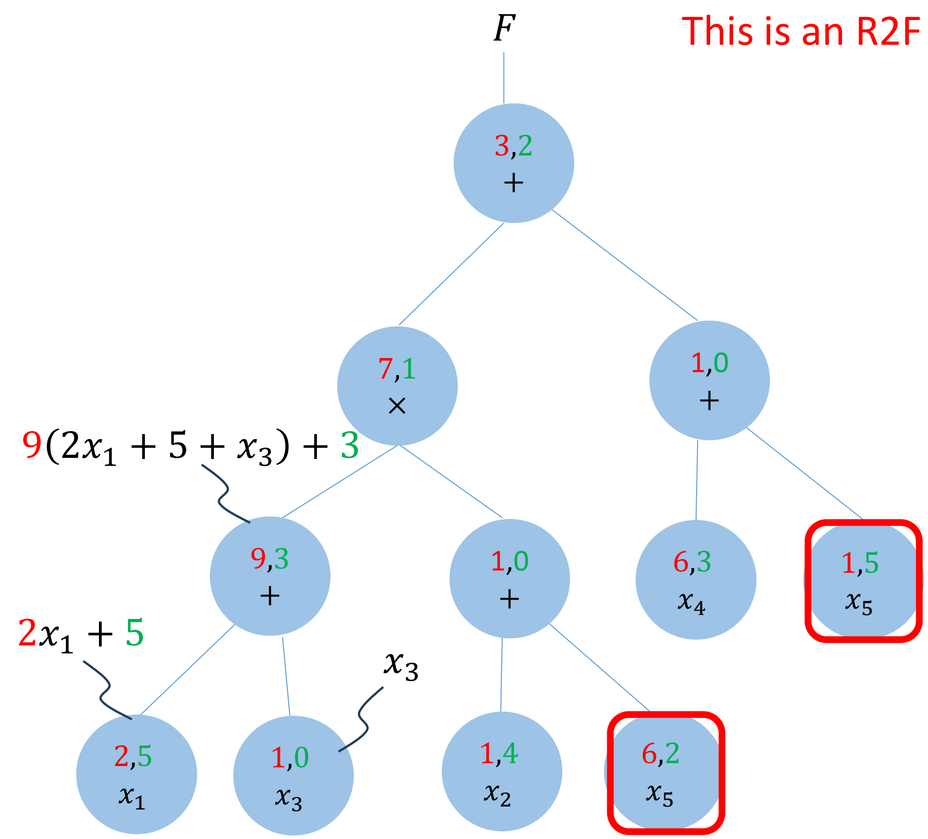

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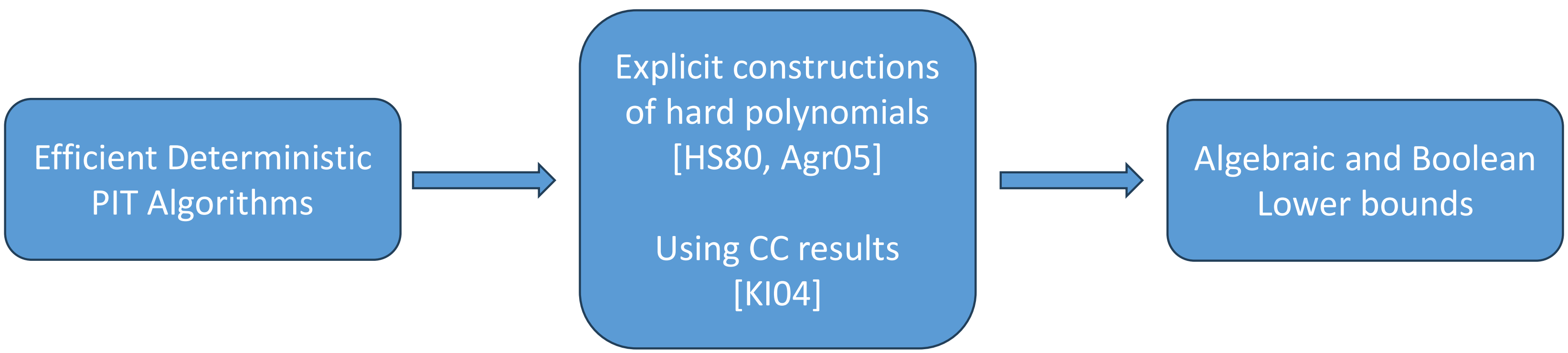
We want to find a **deterministic** algorithm for PIT.



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$f \in \mathbb{F}[\boldsymbol{x}]$ is given to us as a circuit\formula.

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We **hit** $\mathcal{C}\backslash f$ – found a HS for $\mathcal{C}\backslash f$.

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Previous Work

Model	Whitebox	Blackbox	BB Orbits
ROF	Poly	Poly [SV15,MV17]	Quasi-poly [MS22]
R2F	Poly [MRS14]	Quasi-Poly [Pra19, Sha23]	?
R3F	Poly [MRS14]	Quasi-Poly [Pra19]	?
R4F	?	?	?
∑ROF	Poly	Poly [SV15,MV17]	?
Multilinear-RkF	Poly [AvMV15]	Quasi-Poly [AvMV15]	?

[SV15] – Amir Shpilka and Ilya Volkovich (STOC)
[MV17] – Daniel Minahan and Ilya Volkovich (CCC)
[AvMV15] - Anderson et al. (CCC)
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[MS22] – Dori Medini and Amir Shpilka (CCC)
[Sha23] – Nadav Shamir (Master Thesis)
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Previous Work

$$Orbit(\mathcal{C}) = \left\{ f(\ell_1, \dots, \ell_n) \middle| \begin{array}{c} f \in \mathcal{C}, \\ \ell_i\text{'s linearly independent} \end{array} \right\}$$

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Results

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R4F	Poly	Quasi-poly	Quasi-poly
$\sum$ROF	Poly	Poly [SV15,MV17]	Quasi-Poly
Multilinear-RkF	Poly [AvMV15]	Quasi-Poly [AvMV15]	Quasi-poly

* Improved algorithm.

Main Result – PIT for R4Fs

Theorem.

There is a deterministic blackbox PIT algorithm for R4Fs in time:

$$O(2n^{c+18\log n})$$

For constant c.

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What makes read-4 so special?

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PIT for New Model $\xrightarrow[\text{via partial derivatives}]{\text{reduce}}$ Known PIT

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HS' for $\mathcal{C}$ $\xleftarrow{\text{lifting using known results}}$

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$$\text{PIT for New Model} \xrightarrow[\text{via partial derivatives}]{\text{reduce}} \text{Known PIT}$$

- ROF: Homogenization
- $\sum$ROF: PDs + Hardness of Representation → ROF PIT
- R2F: Fragmenting (repeated PDs) → $\sum^2 \text{ROF}$ PIT
- R3F: Fragmenting (repeated PDs) → R2F PIT
- ml-RkF: Fragmenting/Shattering (repeated PDs) → ml-R$(k-1)$F
- Orbits for ROFs: Directional derivatives

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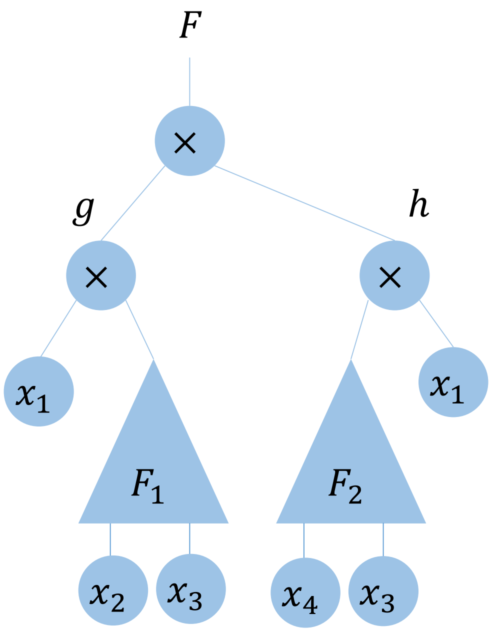
PDs **simplify** the problem

This means, these models are **closed** under PDs

Differentiation Problems



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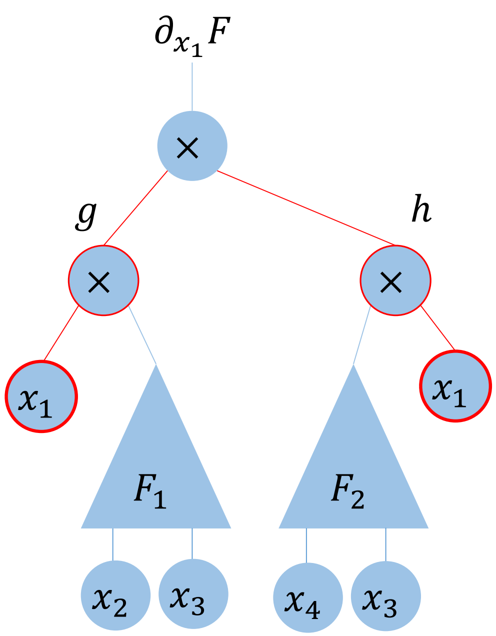

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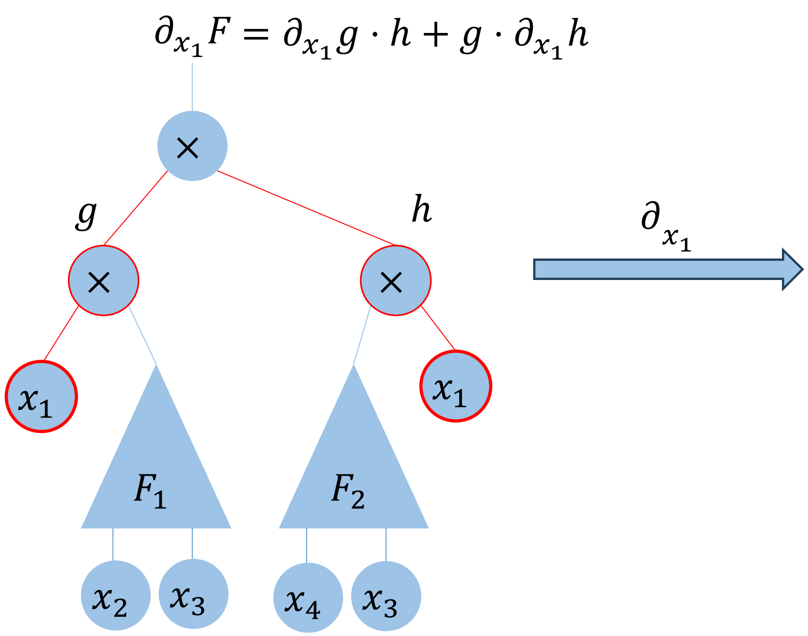

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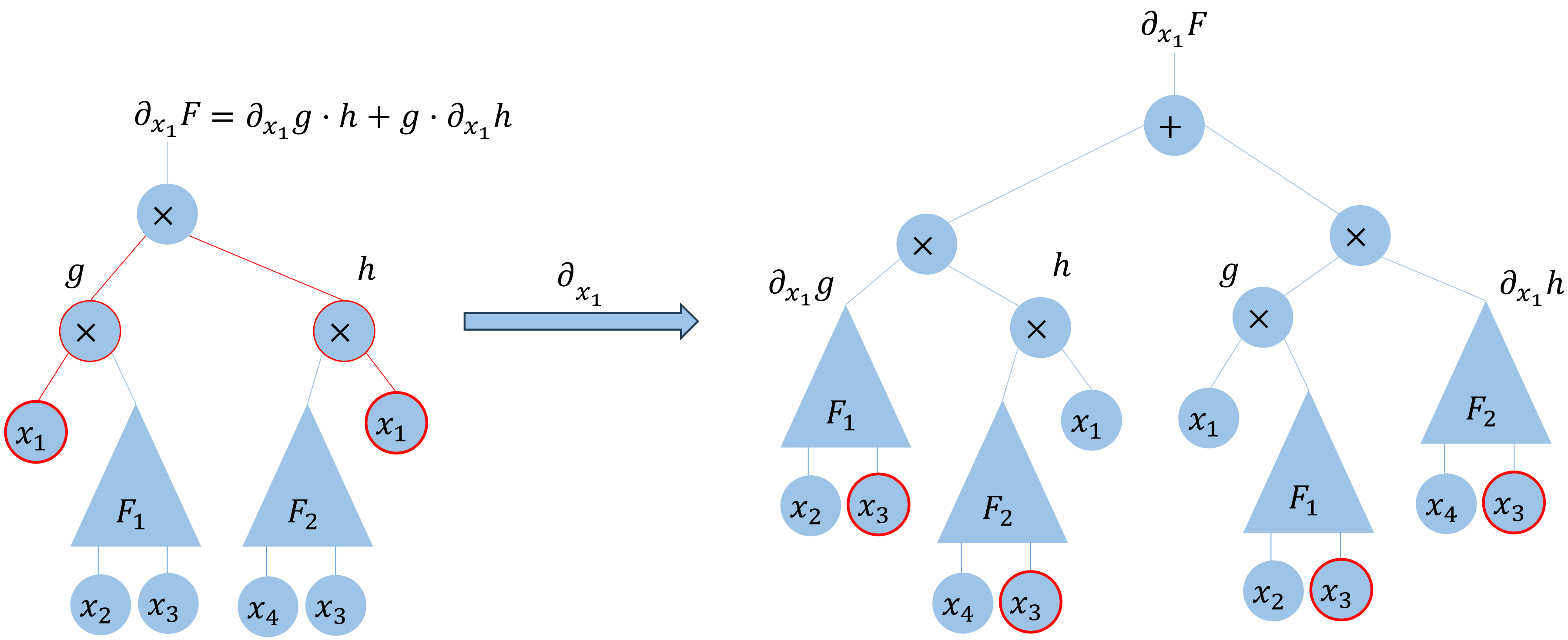

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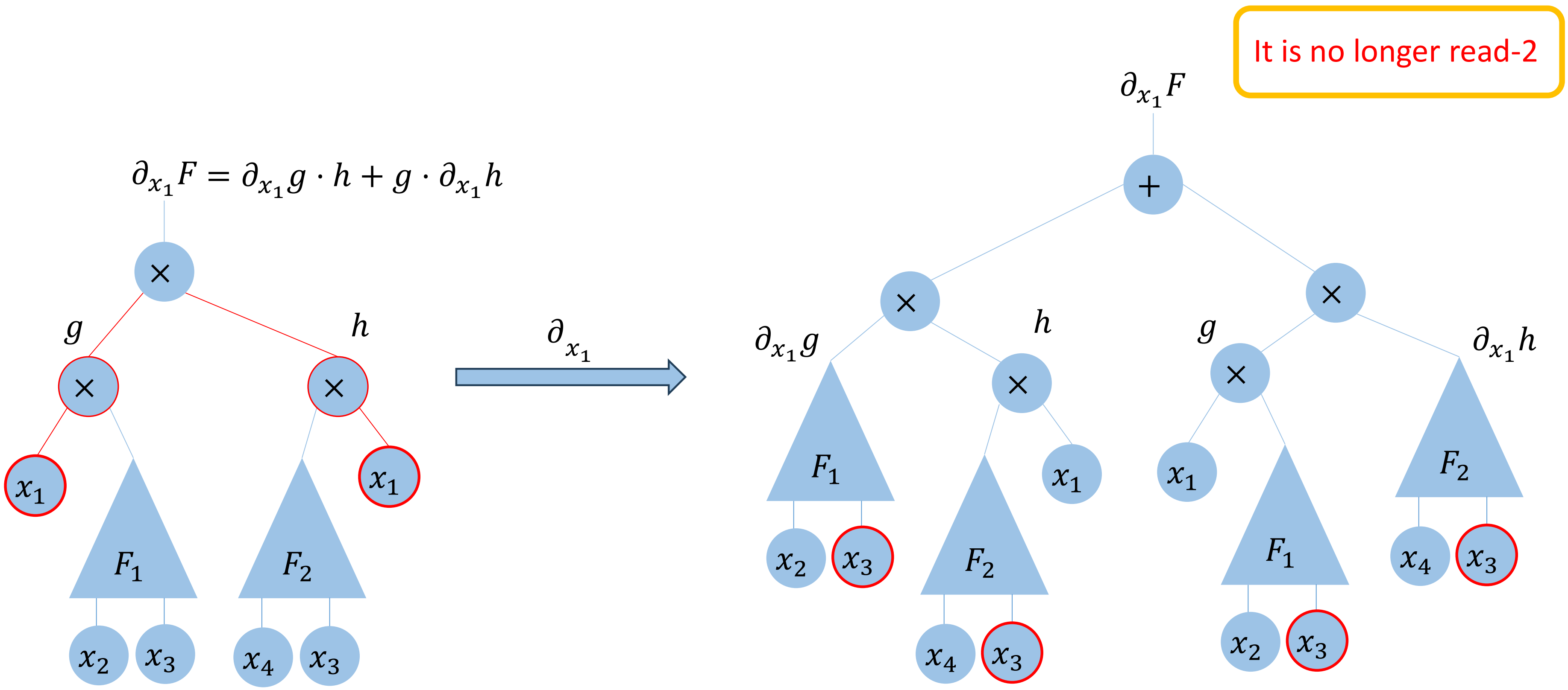
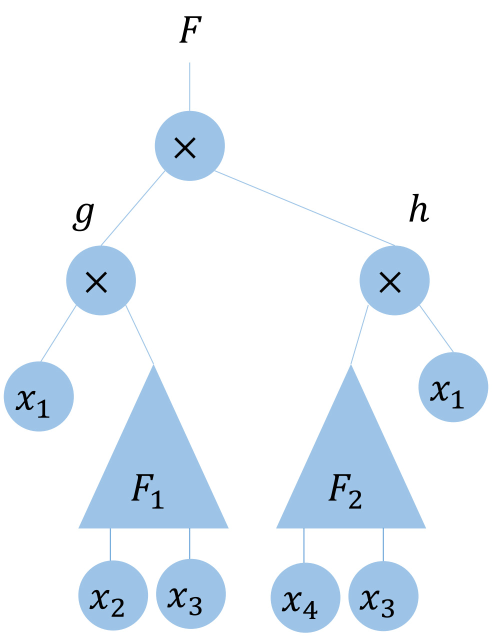


Differentiation Problems

Solution (for R2Fs): Differentiate twice.



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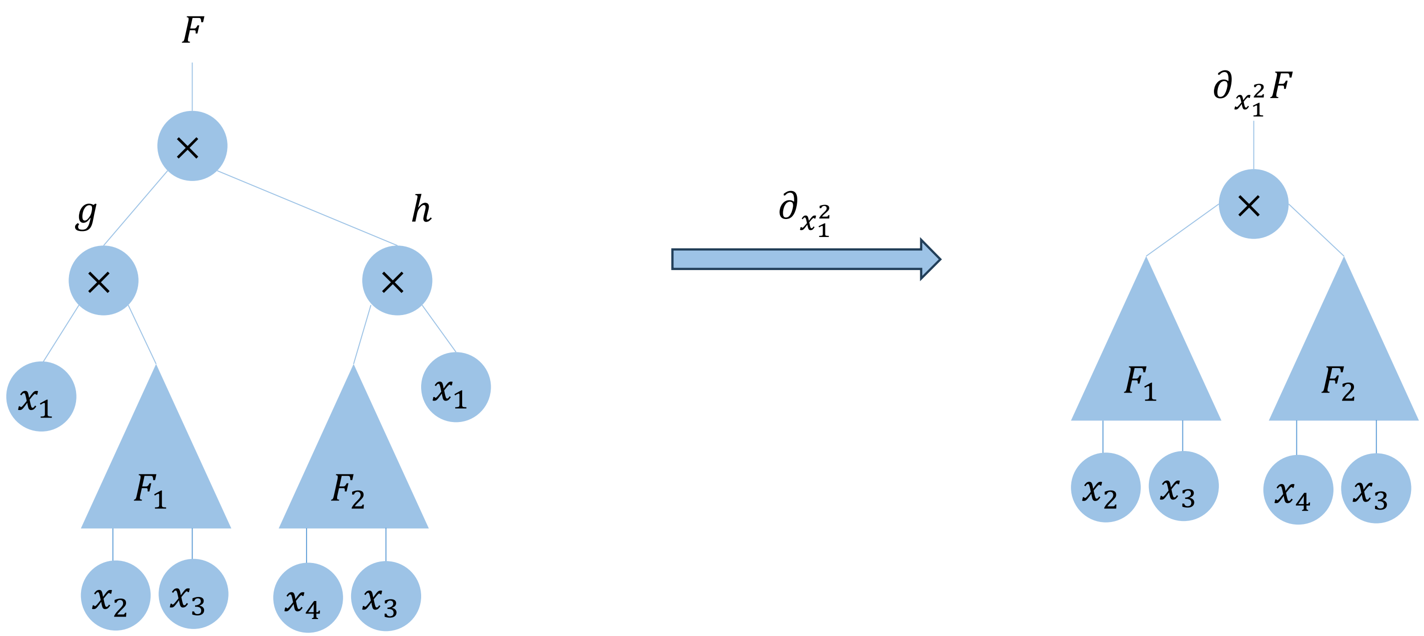

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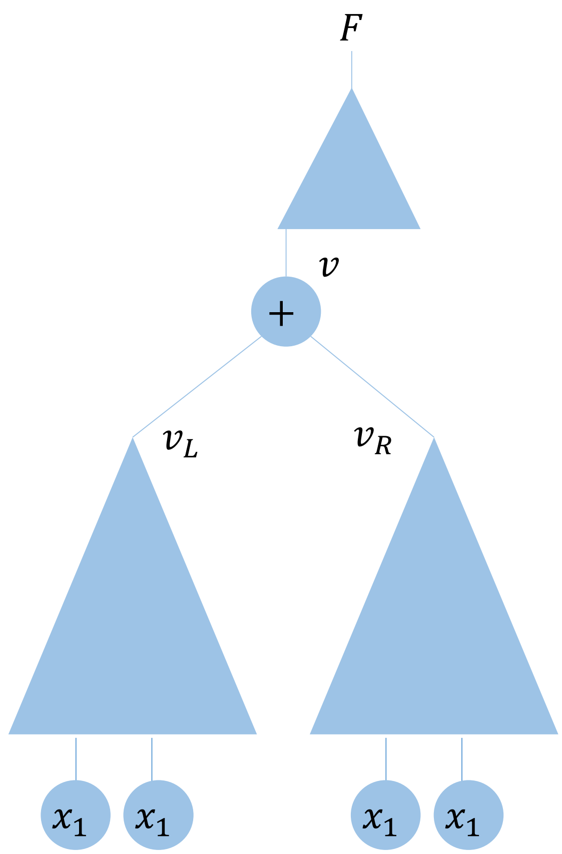

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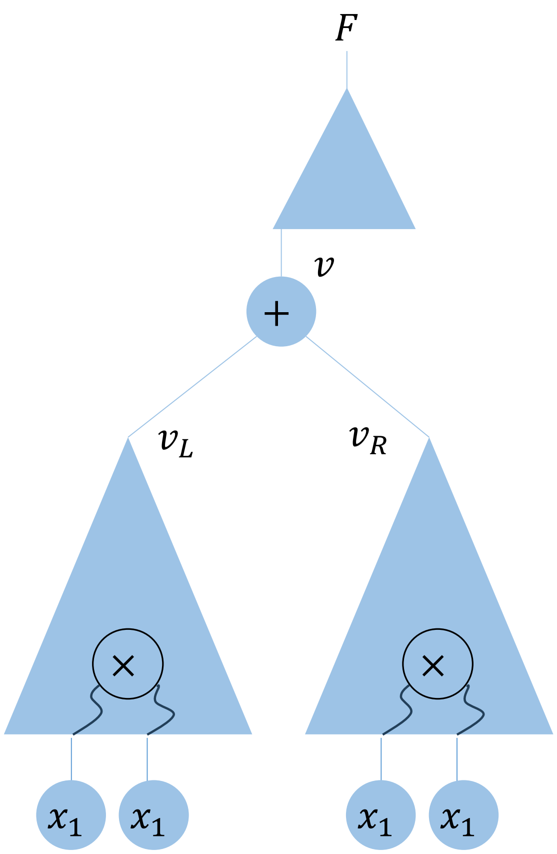

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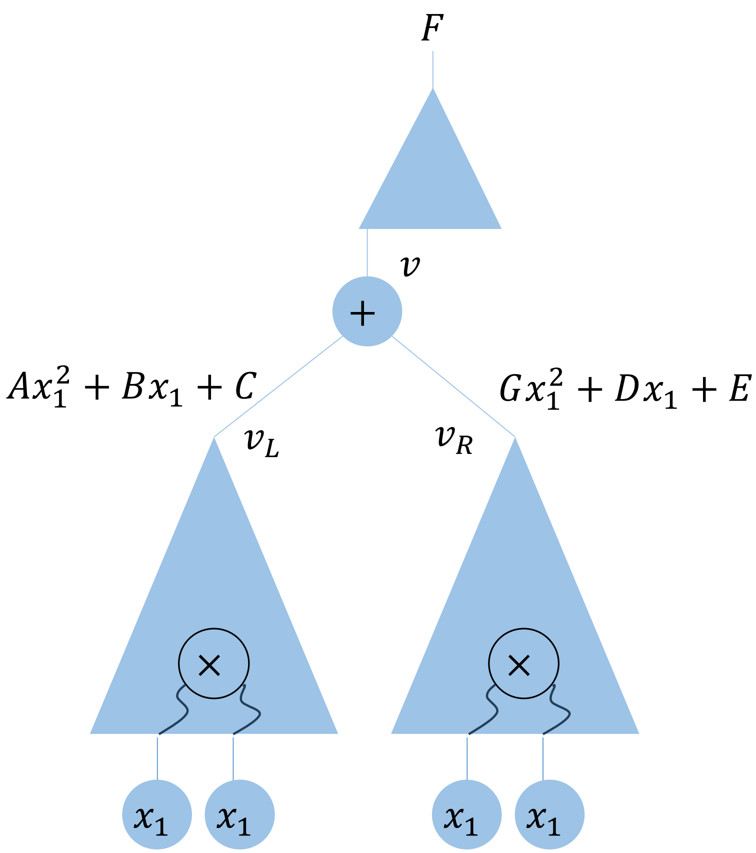

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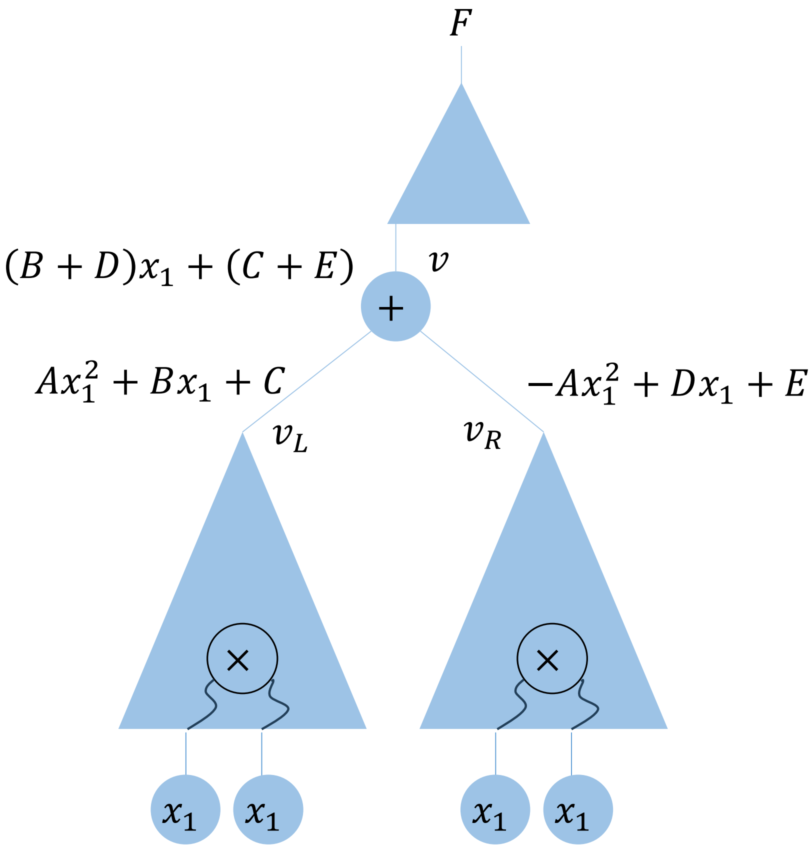
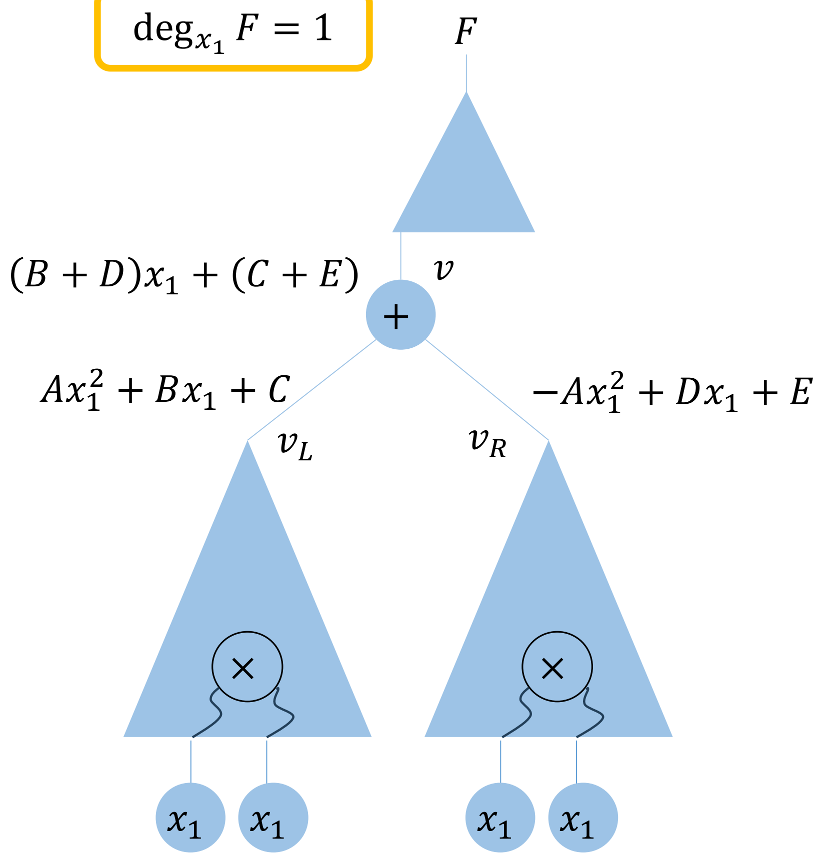


Differentiation Problems

$\boxed{\deg_{x_1} F = 1}$



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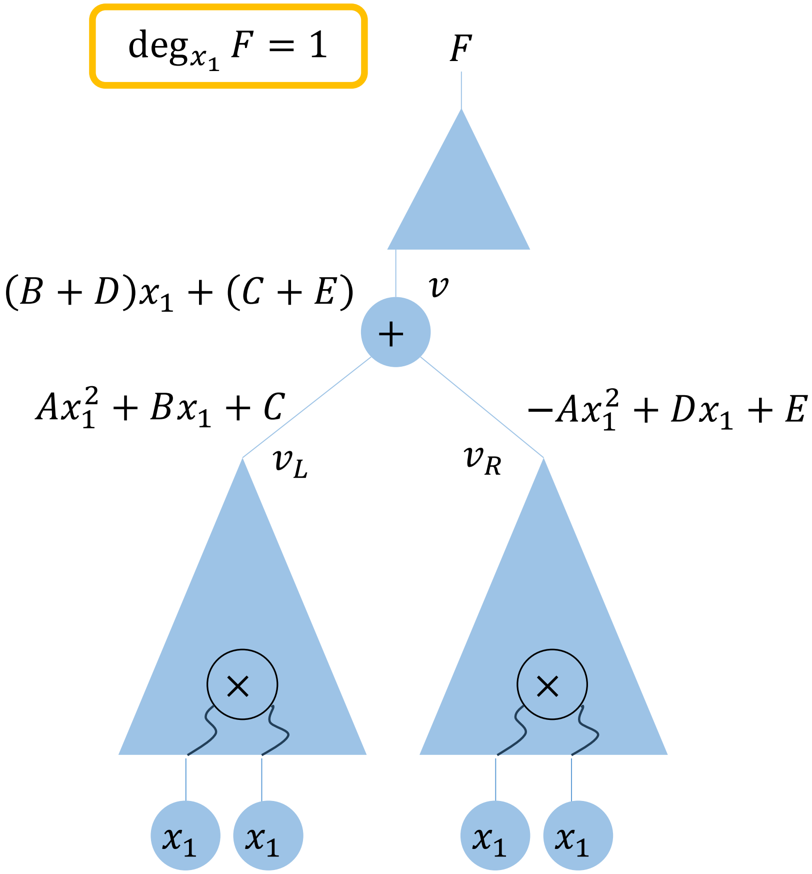

Can we use ∂_{x_1}?

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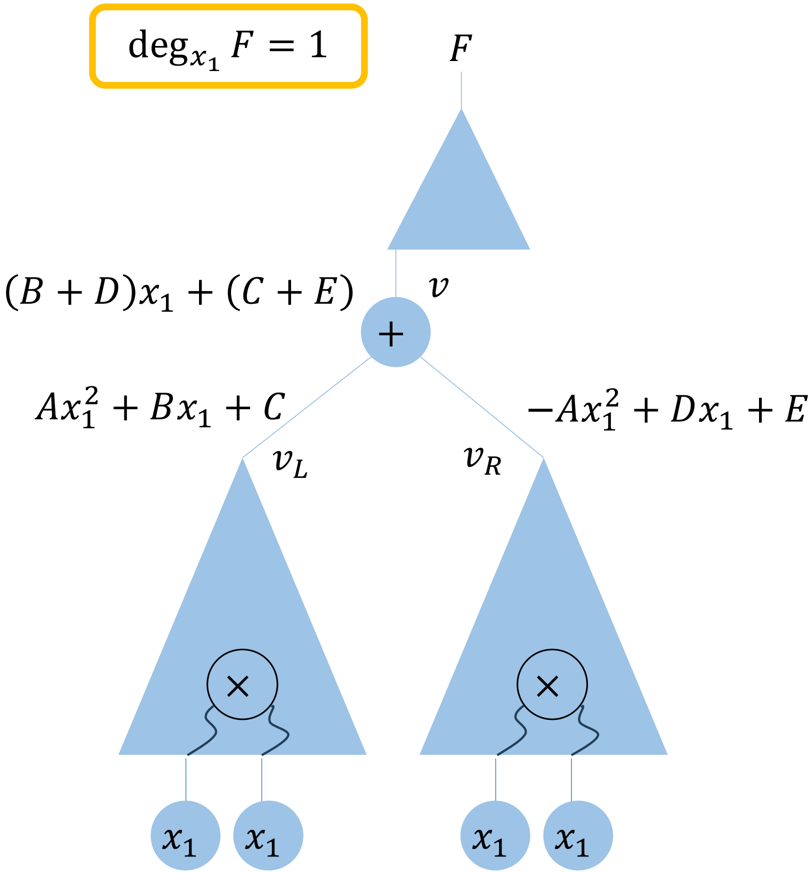

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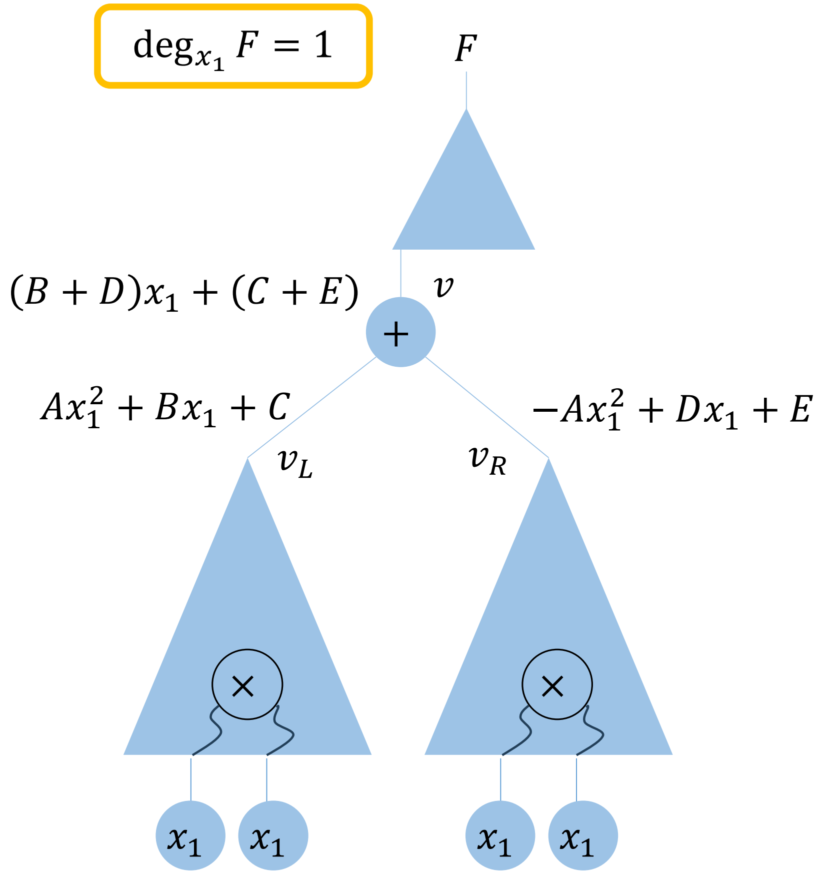

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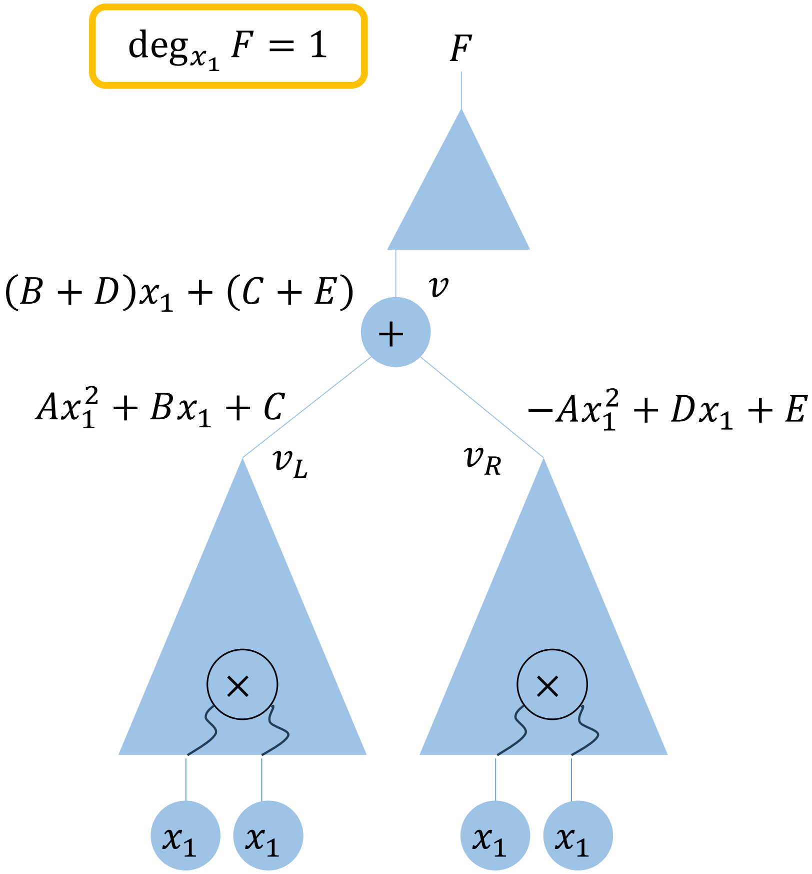

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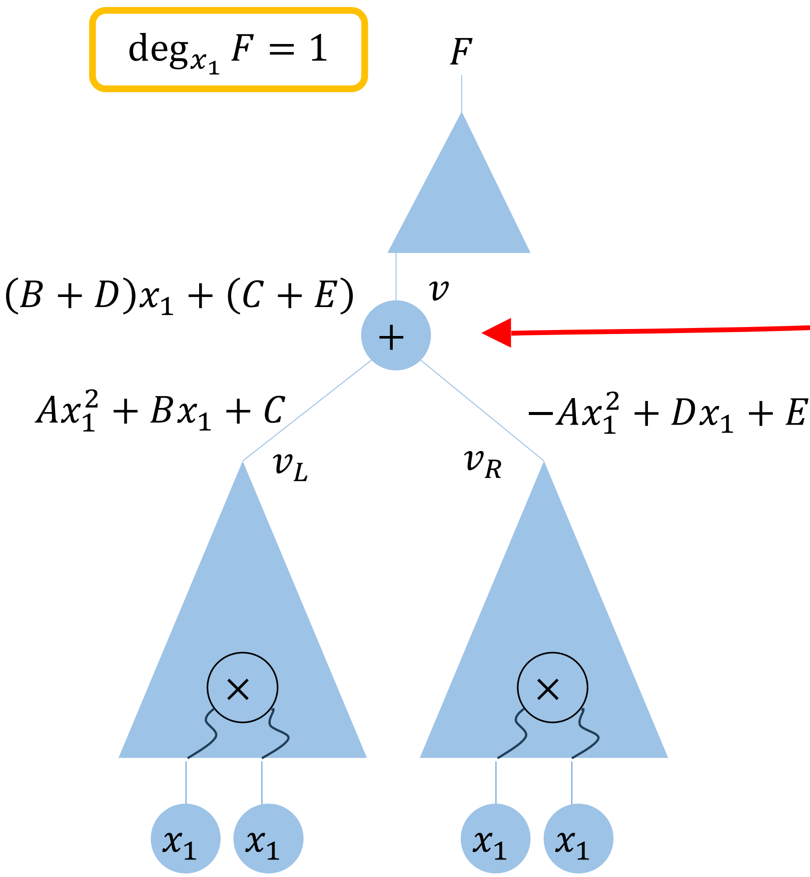

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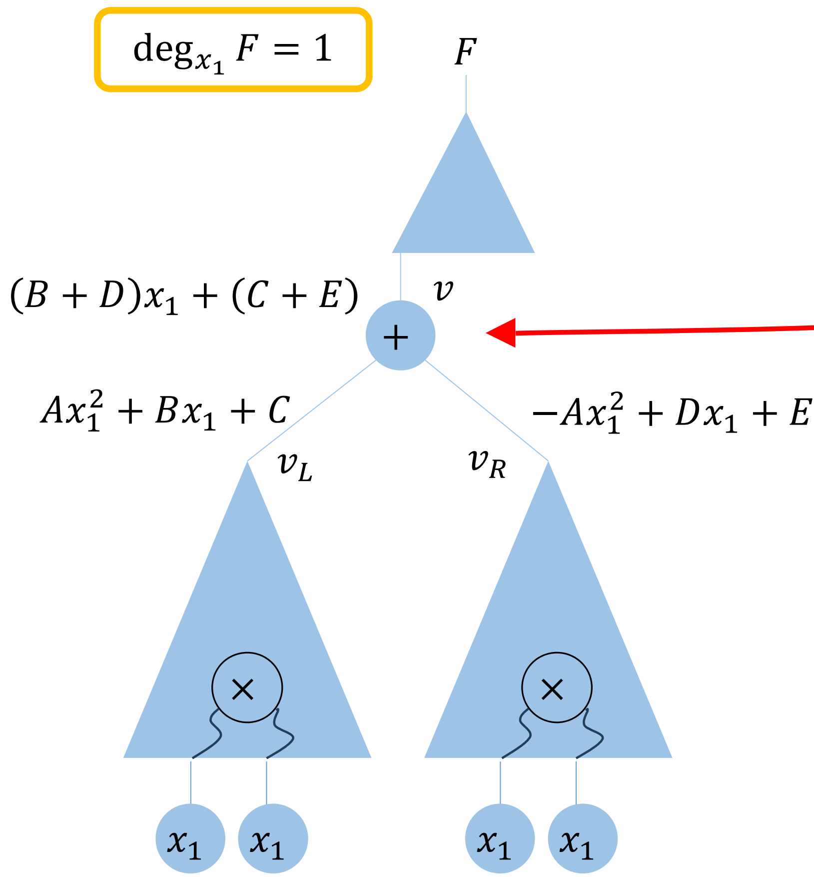

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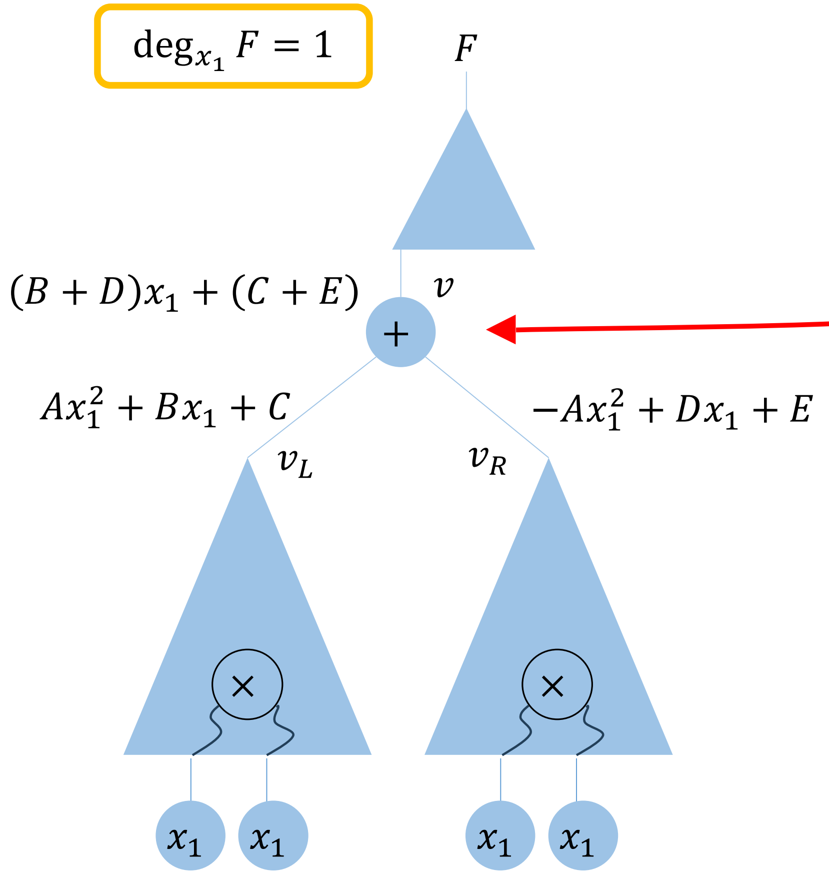

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This is the barrier that breaks the previous methods

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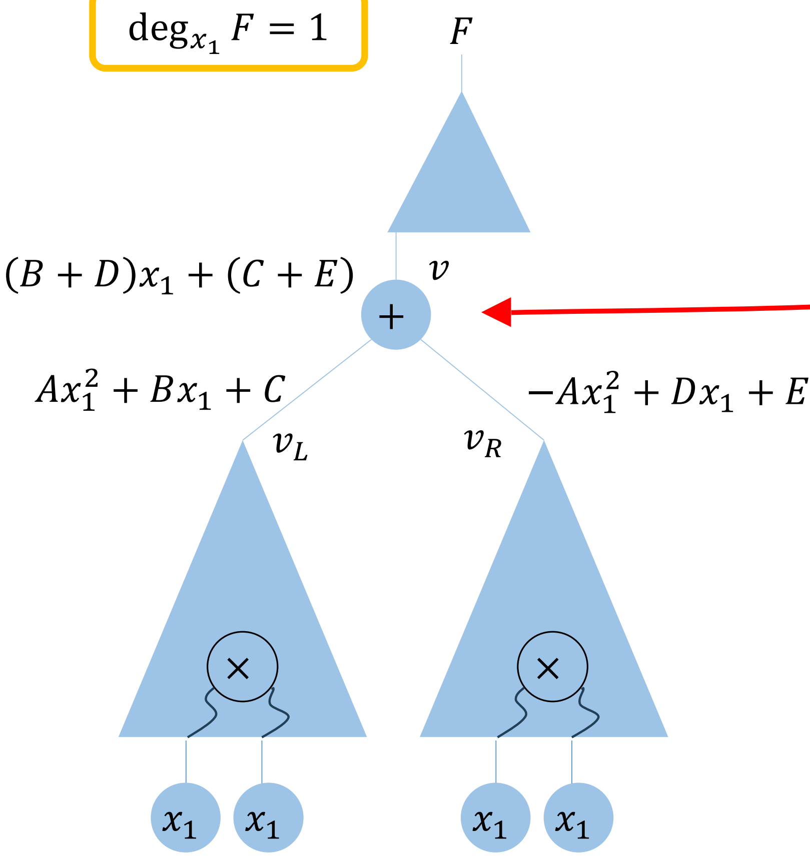

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Cancellations are exactly what makes PIT hard in general

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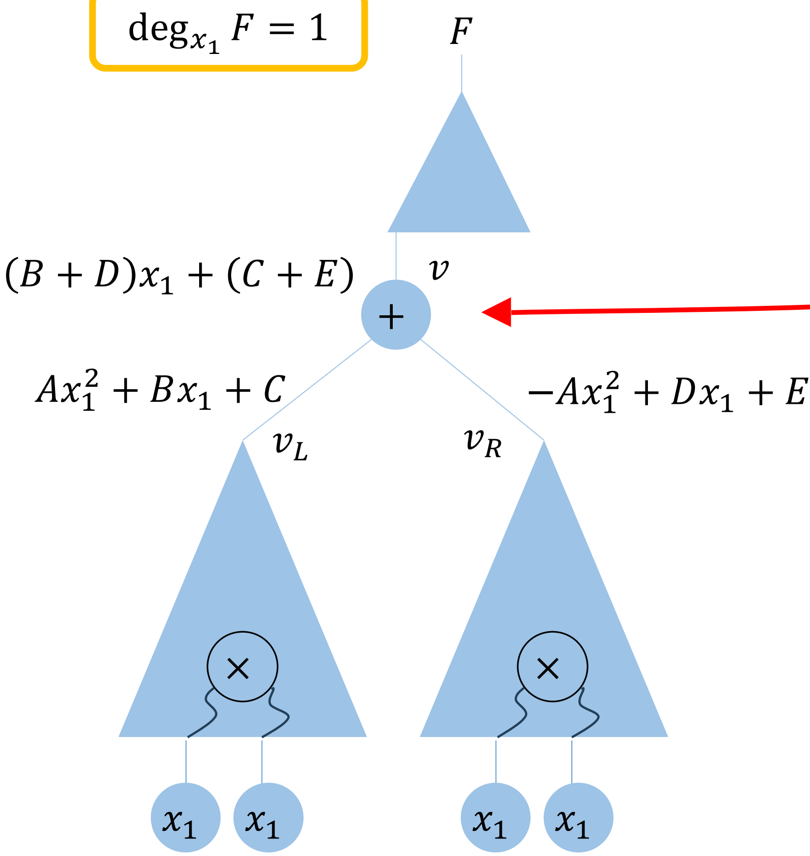

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Can also happen in $\sum^2 \mathrm{R2F}$

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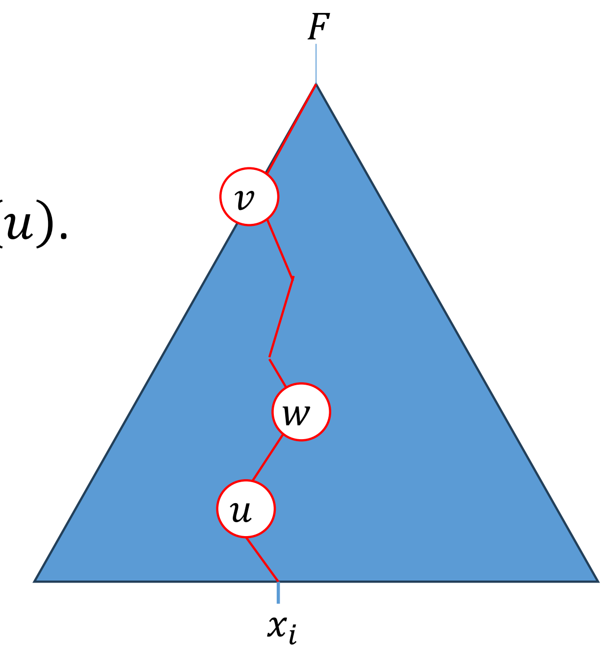
x_i is ***structural*** in F if:

$\forall$ gates v, u: v is an **ancestor** of u $\Rightarrow \deg_{x_i}(v) \geq \deg_{x_i}(u)$.



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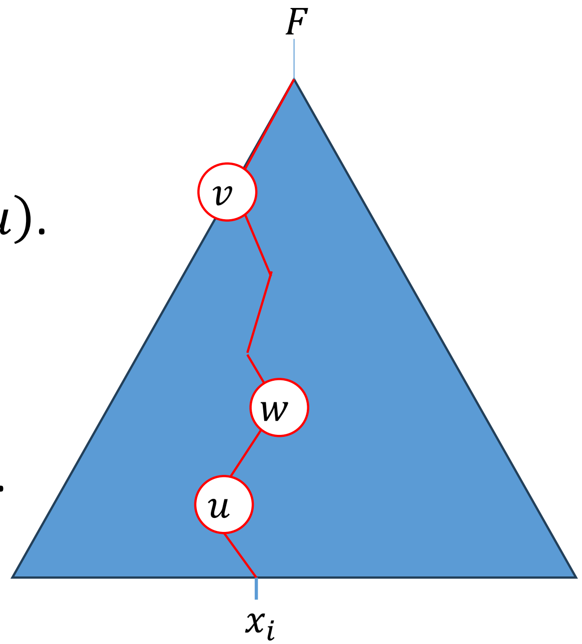

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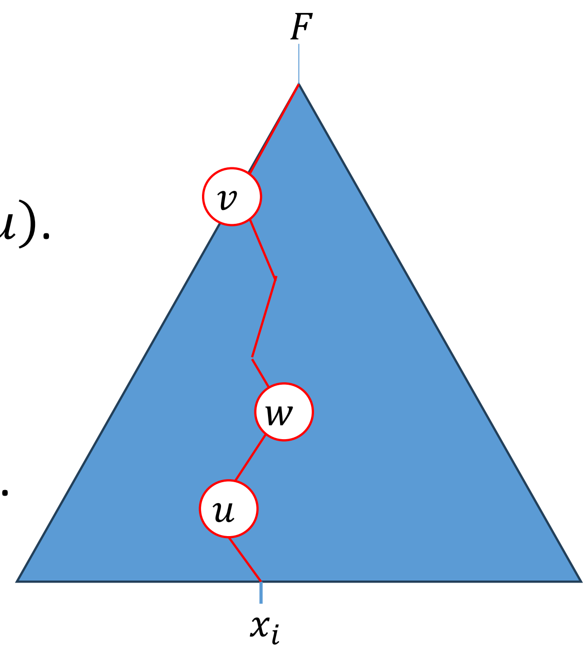
F is **structural** $\Rightarrow$ previous methods **work** (mostly)



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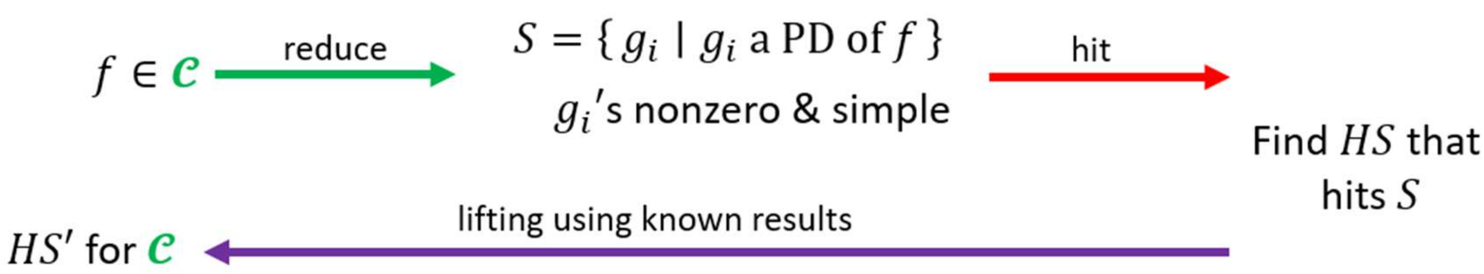

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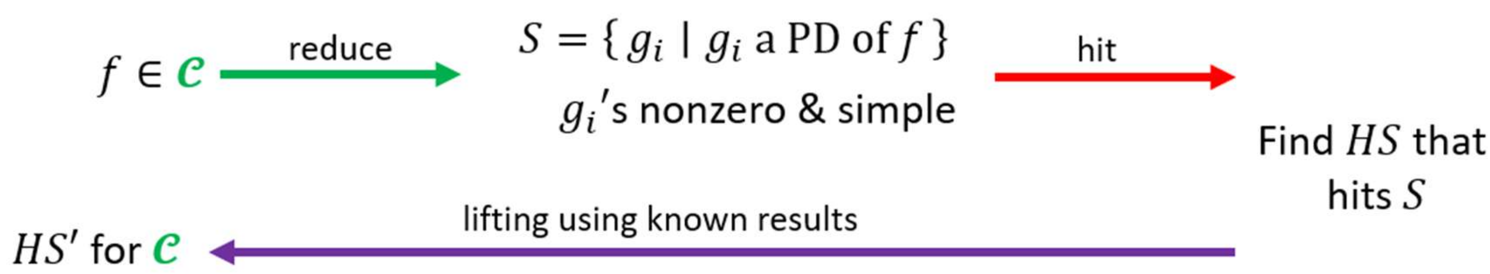

2. Dominating Degree Patterns.

   New closure rules (under taking PDs).

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$f \in \mathcal{C} \xrightarrow{\text{divide}}$ partition $[n] = S_1 \sqcup \cdots \sqcup S_k$ $\xrightarrow{\text{conquer}}$ find efficient PIT for every $f|_{S_i}$

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More structure

gives efficient PIT for $\mathcal{C}$ $\xleftarrow{\text{using our new lemma}}$ find efficient PIT for every $f|_{S_i}$

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2. $f|_{S_2} \in$ "Totally non-structural" $\sum^2 \mathrm{R2F}$ - strong structural constraints.

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Old closure lemmas:

$$f \in \mathrm{R}k\mathrm{F} \quad \Rightarrow \quad \partial_{x^d}(f) \in \mathrm{R}k\mathrm{F}$$

This is local: **one** variable at a time.

Dominating Degree Patterns

Gives more closure rules under PDs.

Old closure lemmas:

$$f \in \mathrm{R}k\mathrm{F} \quad \Rightarrow \quad \partial_{x^d}(f) \in \mathrm{R}k\mathrm{F}$$

This is local: **one** variable at a time.

New idea:

What about global conditions involving **several** variables?

Choose $J \subseteq [n]$ and differentiate by a maximal exponent vector on J.

Dominating Degree Patterns - Example

$$f(x, y, z, w) = (5zw^3)xy^2 + (2z^2w)x^3y^2 + (5)x^2y^2 + (zw)x + (7w)y^3$$

Take $J = \{x, y\}$;

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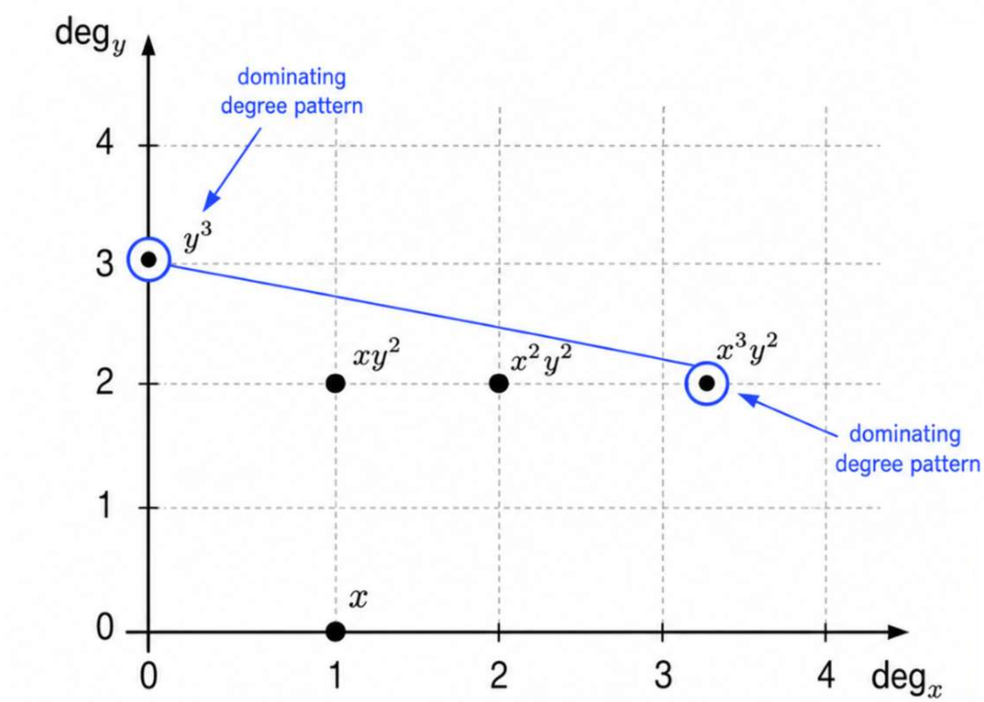

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Let $f \in \mathrm{R2F}$.

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$\vec{e}$ dominating in $f \quad \Rightarrow \quad \partial_{\vec{e}} f \in \mathrm{R2F}$.

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Let $f \in \mathrm{R2F}$.

Let $J \subseteq [n]$ of at most 3 variables.

Let $\vec{e} \in \mathbb{N}^J$.

Current limitations:

- Variables in J are read-2
- Size of J is at most 3

$$\vec{e} \text{ dominating in } f \quad \Rightarrow \quad \partial_{\vec{e}} f \in \mathrm{R2F}.$$

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1. Can a dominating degree pattern lemma be shown for general RkFs and general set size?

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Can our techniques be used for higher reads? What about R5Fs?

More concretely:

1. Can a dominating degree pattern lemma be shown for general RkFs and general set size?
2. Can a better Hardness of Representation result be shown for Resultants of ROFs?

Thanks!