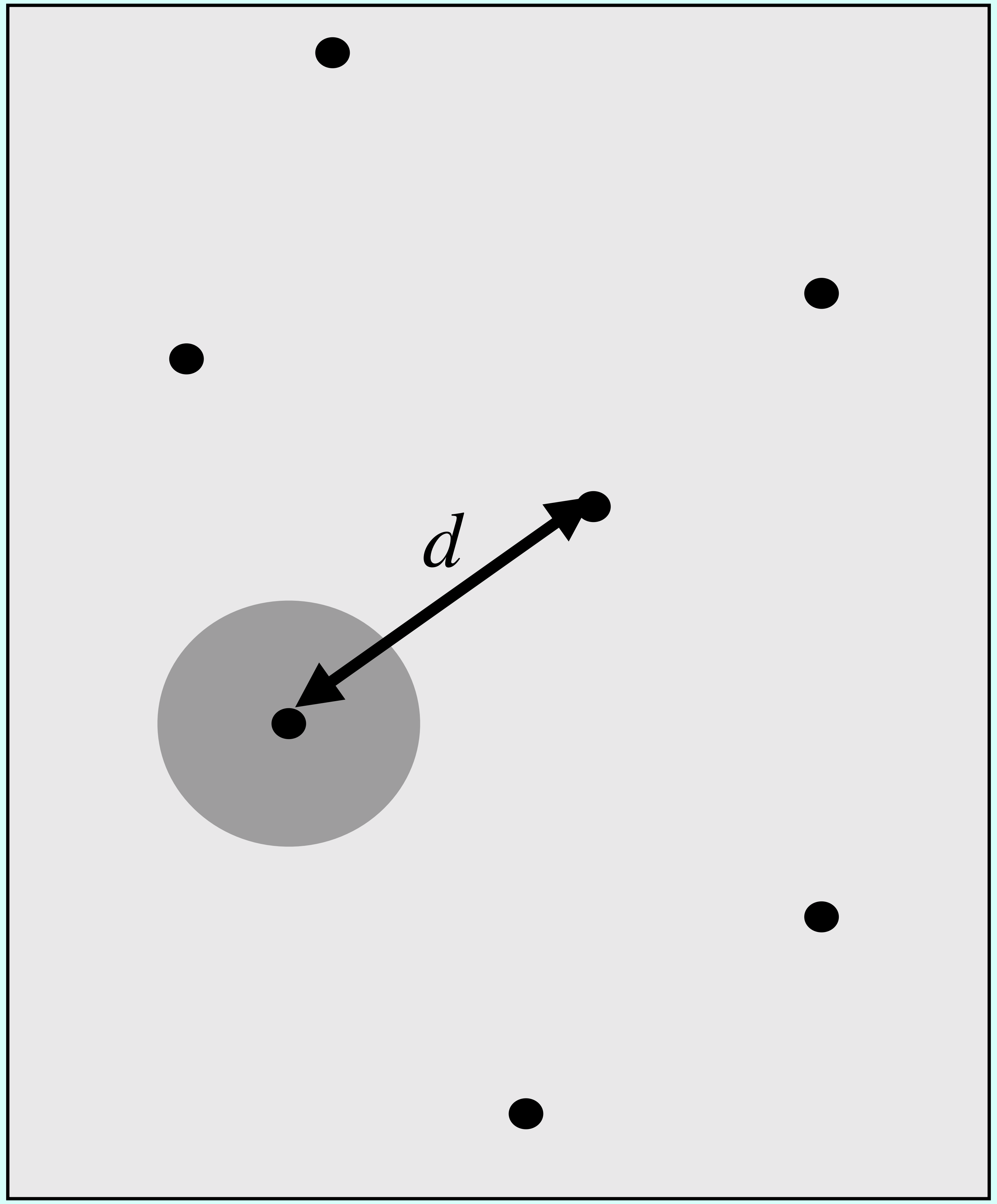
Tight Lower Bound for Approximating Parametrized Maximum Likelihood Decoding under ETH

Rishav Gupta, Bingkai Lin and Xin Zheng

Basics of Coding Theory

What are Codes?



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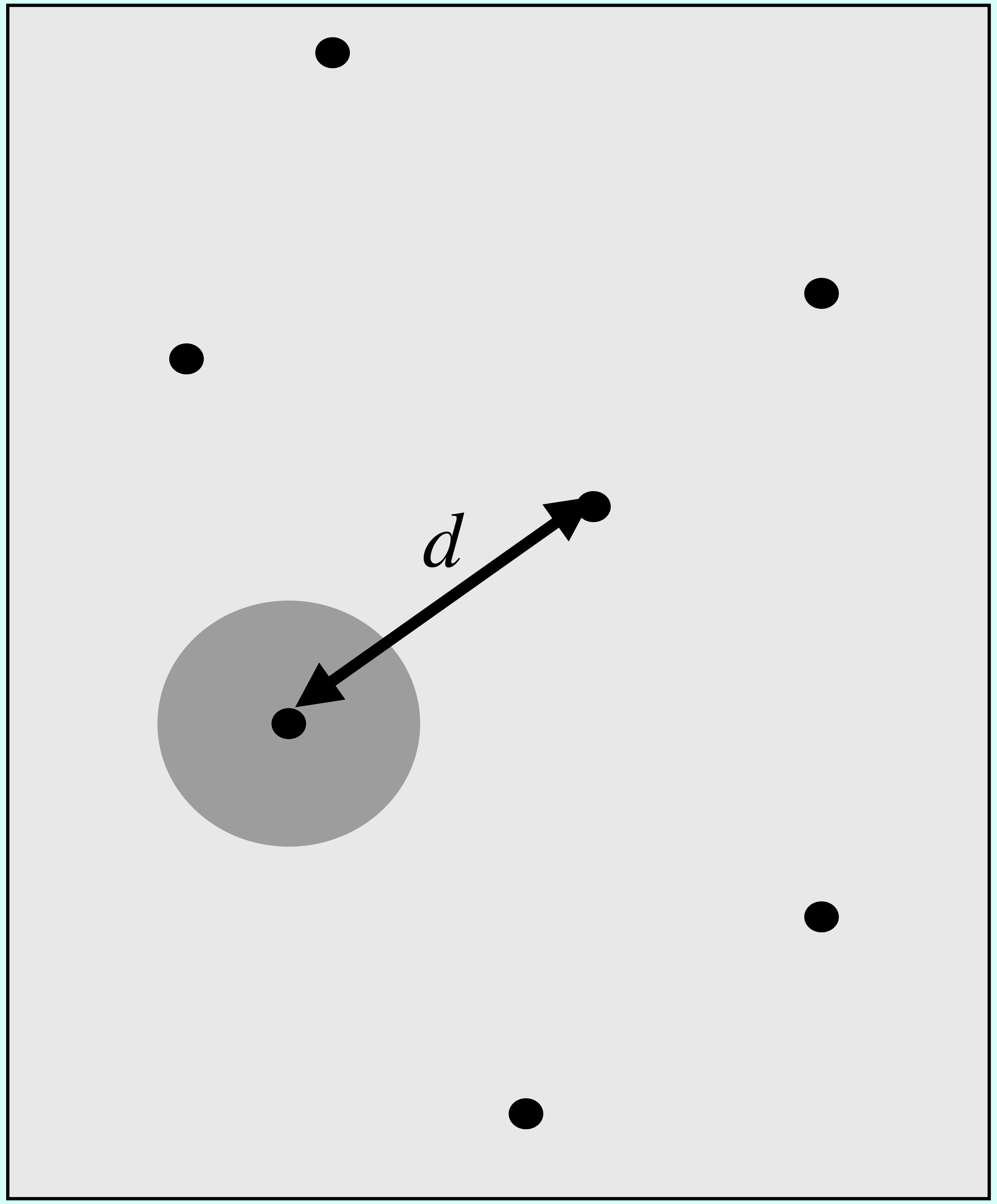

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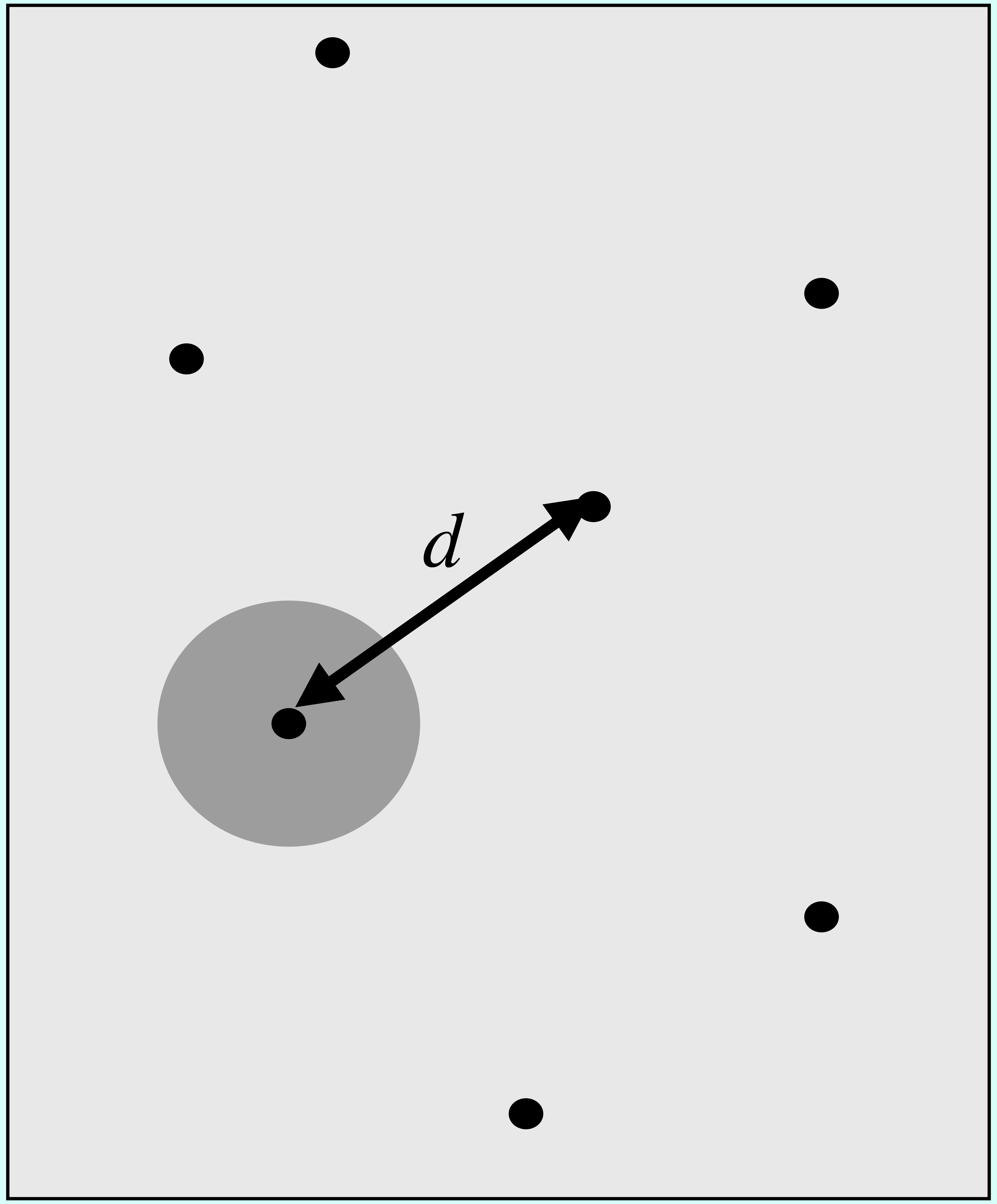

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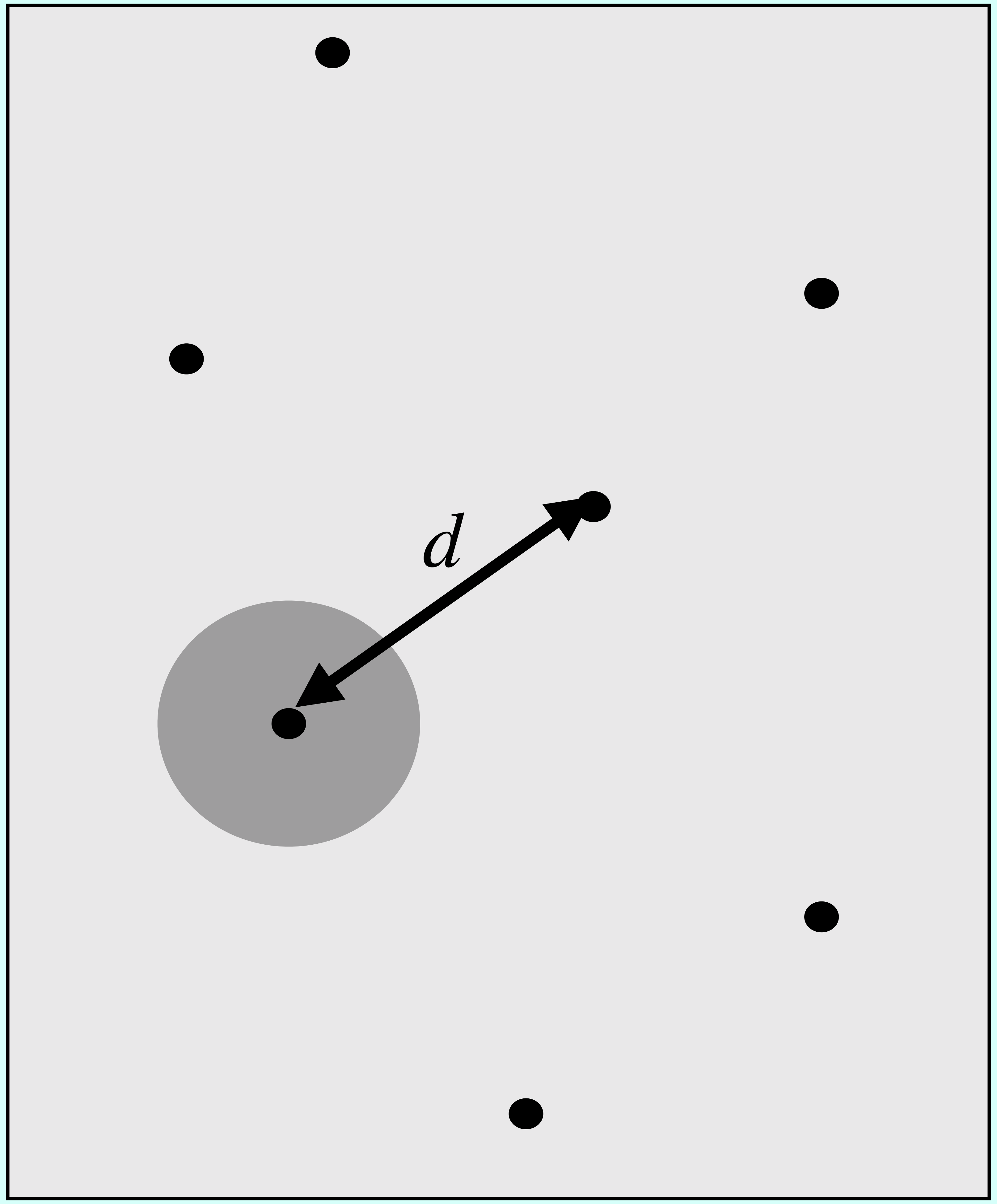

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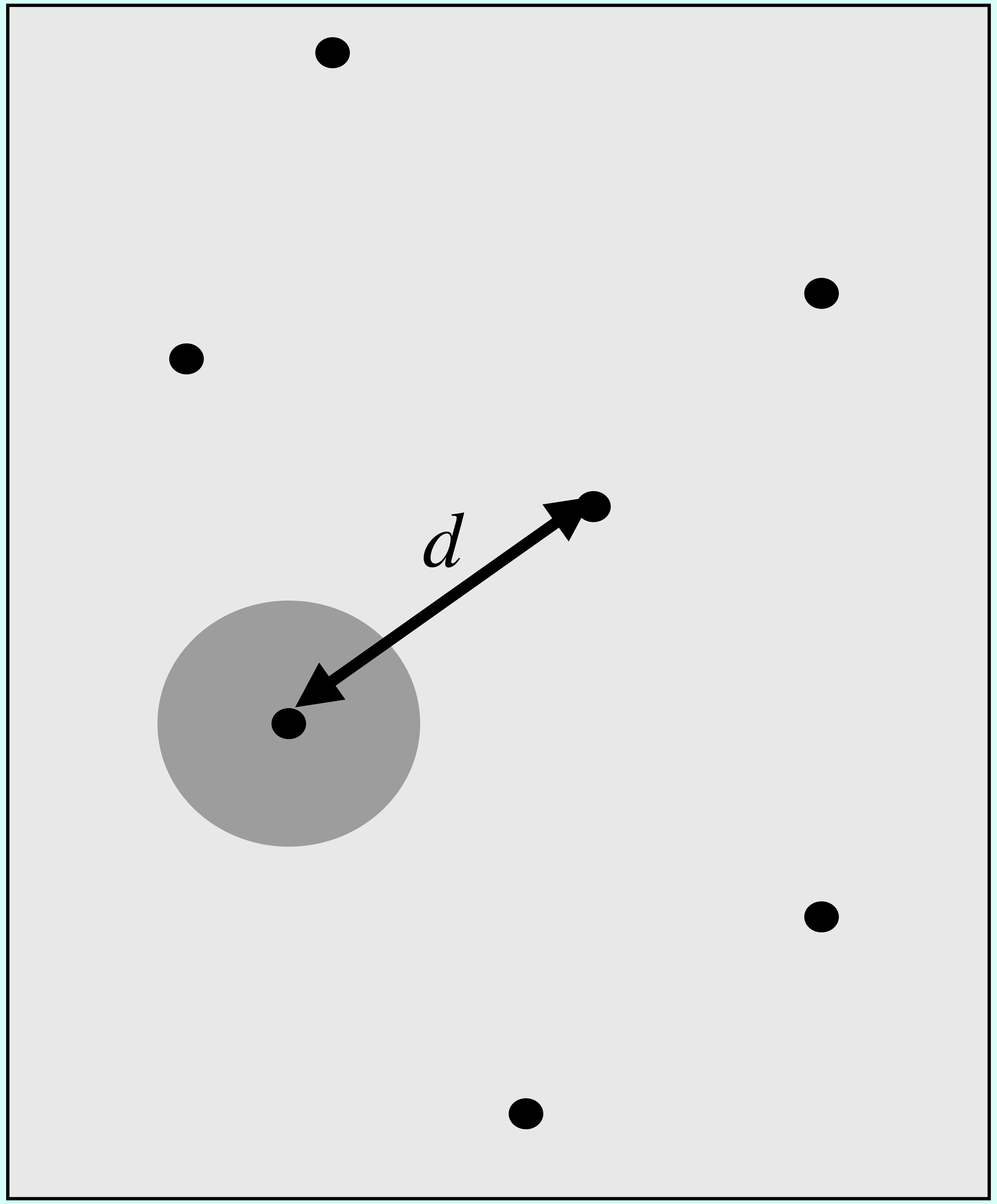

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- **YES** Case: $\mathrm{dist}(y, \mathrm{code}(A)) \leq d.$
- **NO** Case: $\mathrm{dist}(y, \mathrm{code}(A)) > \gamma d.$

Parameterized Gap-NCP

Gap-k-NCP: Given y and A distinguish between,

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Parameterized Gap-MLD

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NCP $\Longleftrightarrow$ MLD

$$A, b = Ax + e$$

$$\downarrow$$

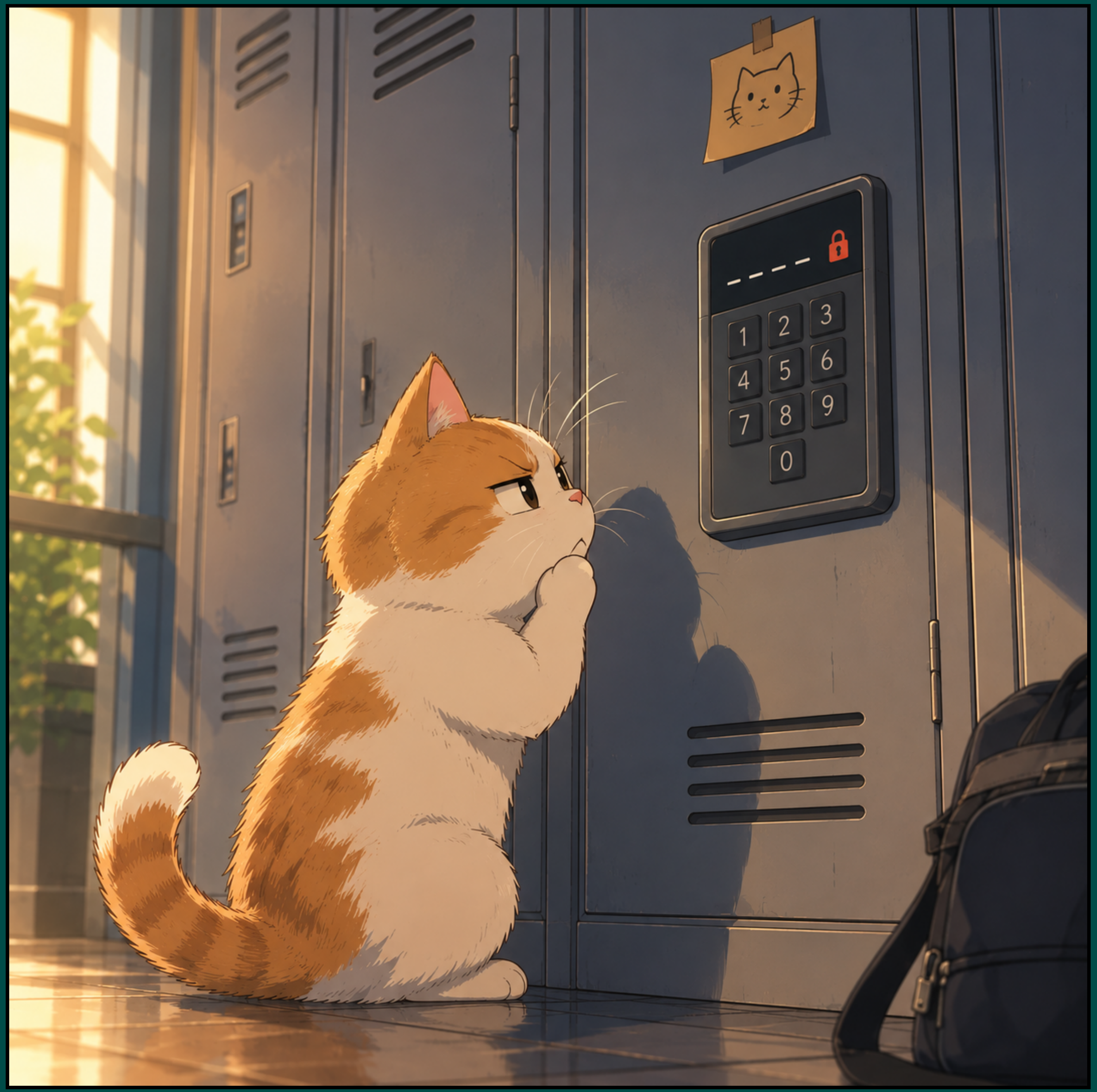
$$H, c := Hb = H \cdot Ax + He$$

Since $H \cdot A = 0$

$$\downarrow$$

$$H, He = c$$

How to Decode?



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- Hence the best brute force algorithm will switch at

$$k = O\left(\frac{n}{\log m}\right).$$

Hardness Assumptions

Exponential Time Hypothesis: ETH-[IP01]

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Consider an instance of 3 SAT,

$$\Phi = C_1 \wedge C_2 \wedge \cdots \wedge C_m, \qquad C_j = z_{j,1} \vee z_{j,2} \vee z_{j,3}, \qquad z_{j,k} \in \{x_i, x_i^c \mid i \in [n]\}$$

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- **ETH** (randomized) is same as **ETH** but also infers all randomized algorithms.

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A reduction from 3SAT with n_0 variables to Gap-NCP instance $A_{m \times n}, b$ over $\mathbb{F}_q$ with $n = \mathcal{O}(n_0)$, $m = \mathcal{O}(n_0)$, and some constants $\alpha, \gamma > 1$.

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Our Results


Study
Hard
Be
Kind
MATHEMATICS
PHYSICS
CHEMISTRY
BIOLOGY
HISTORY

Hardness of Gap-k-NCP/MLD

Result	Assumption	Lower bound	Approx. factor
[Man20]	randomized Gap-ETH	No $m^{o(k)}$ algorithm	all constant factors
[BKM25]	ETH	No $m^{o(k/\log^c k)}$ algorithm	all constant factors
Our result	ETH	No $m^{o(k)}$ algorithm	some constant factor
Our result (informal)	randomized ETH	Brute-force optimal; no $m^{o(k)}$ even when $k \leq m^{\varepsilon}$	some constant factor

Reduction Sketch

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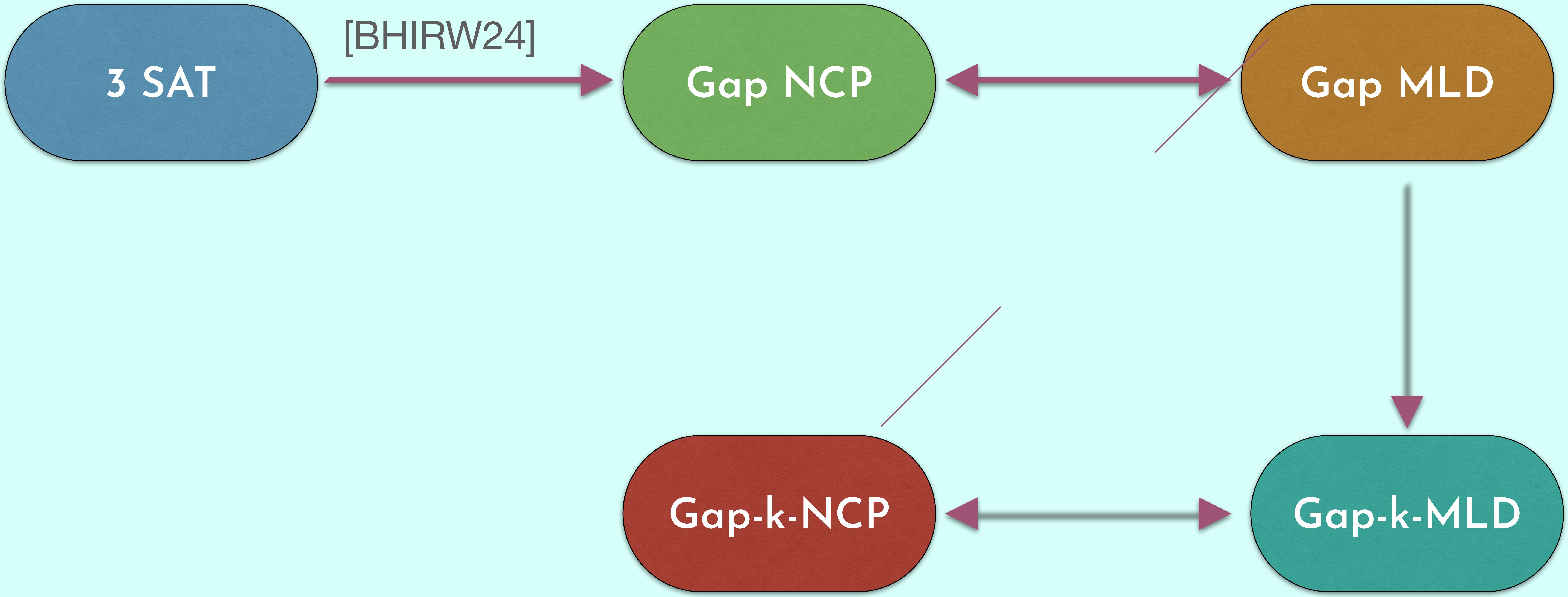

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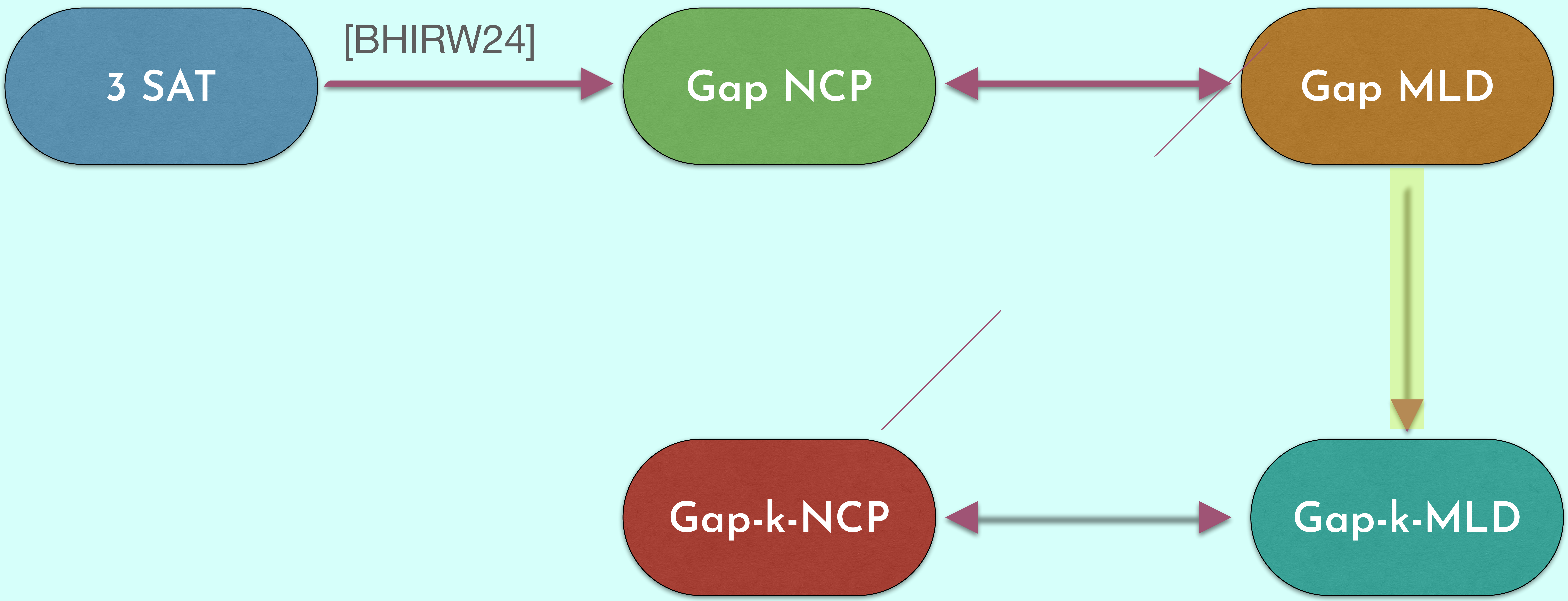

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$$\underbrace{\begin{bmatrix} | & | & & | \\ h_1 & h_2 & \cdots & h_m \\ | & | & & | \end{bmatrix}}_{H \in \mathbb{F}_q^{m-n \times m}} \cdot e = c,$$

- **YES Case:** c is in span of some αm $h_i's$.
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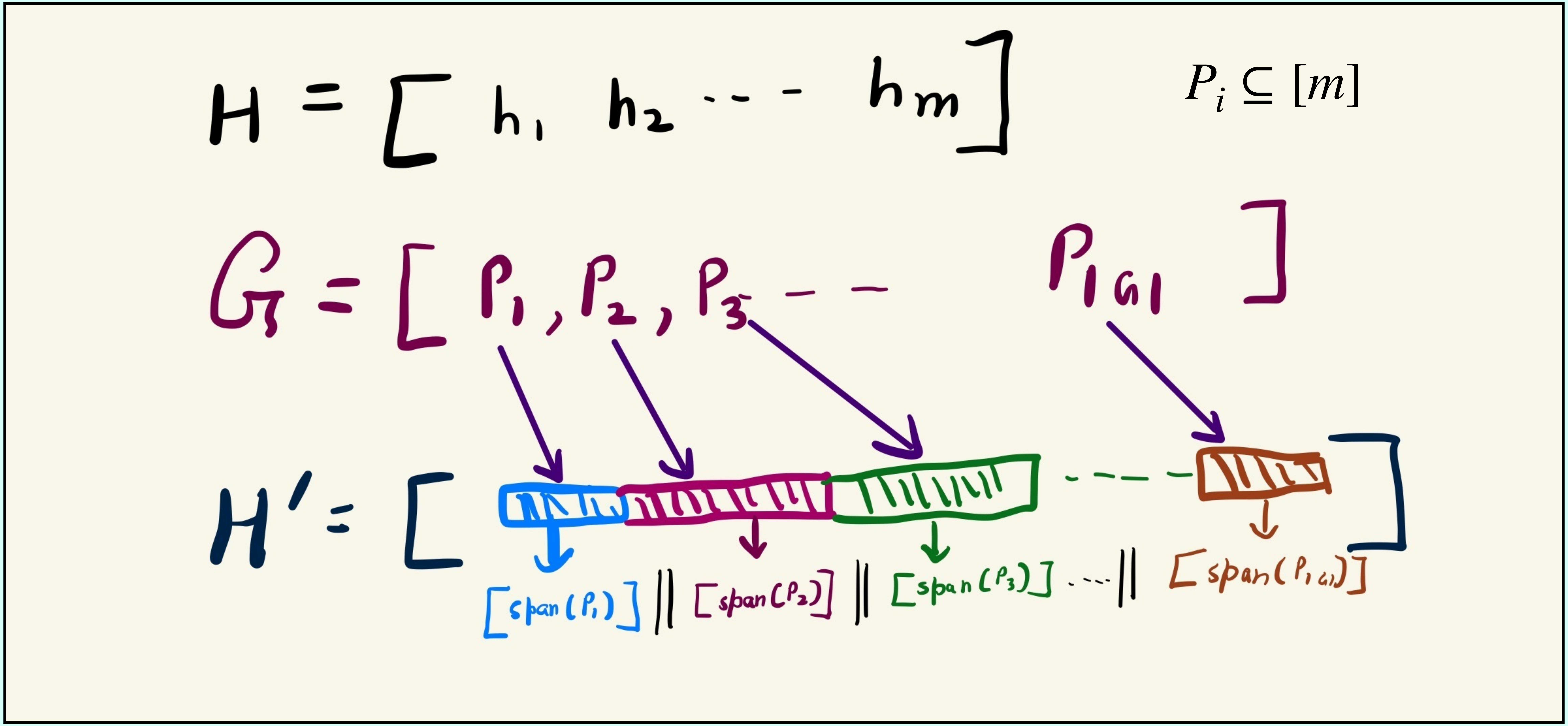
- To get parametrized hardness we would want to take the above as input, and a parameter k and generate H' and c' such that,
  - **YES Case:** c' is in the span of some k columns of H'.
  - **NO Case:** c' is not in the span of any γk columns of H'.

Grouping Reduction



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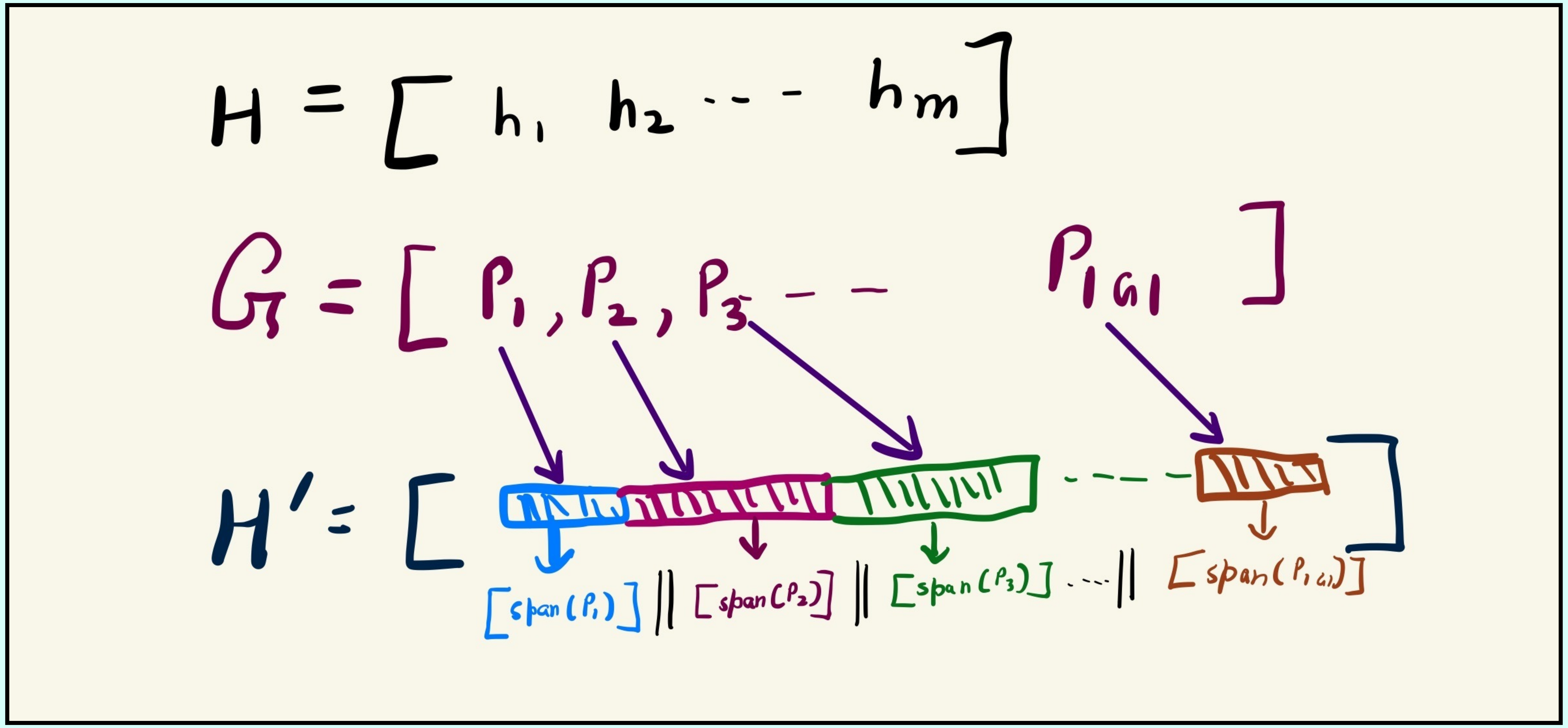

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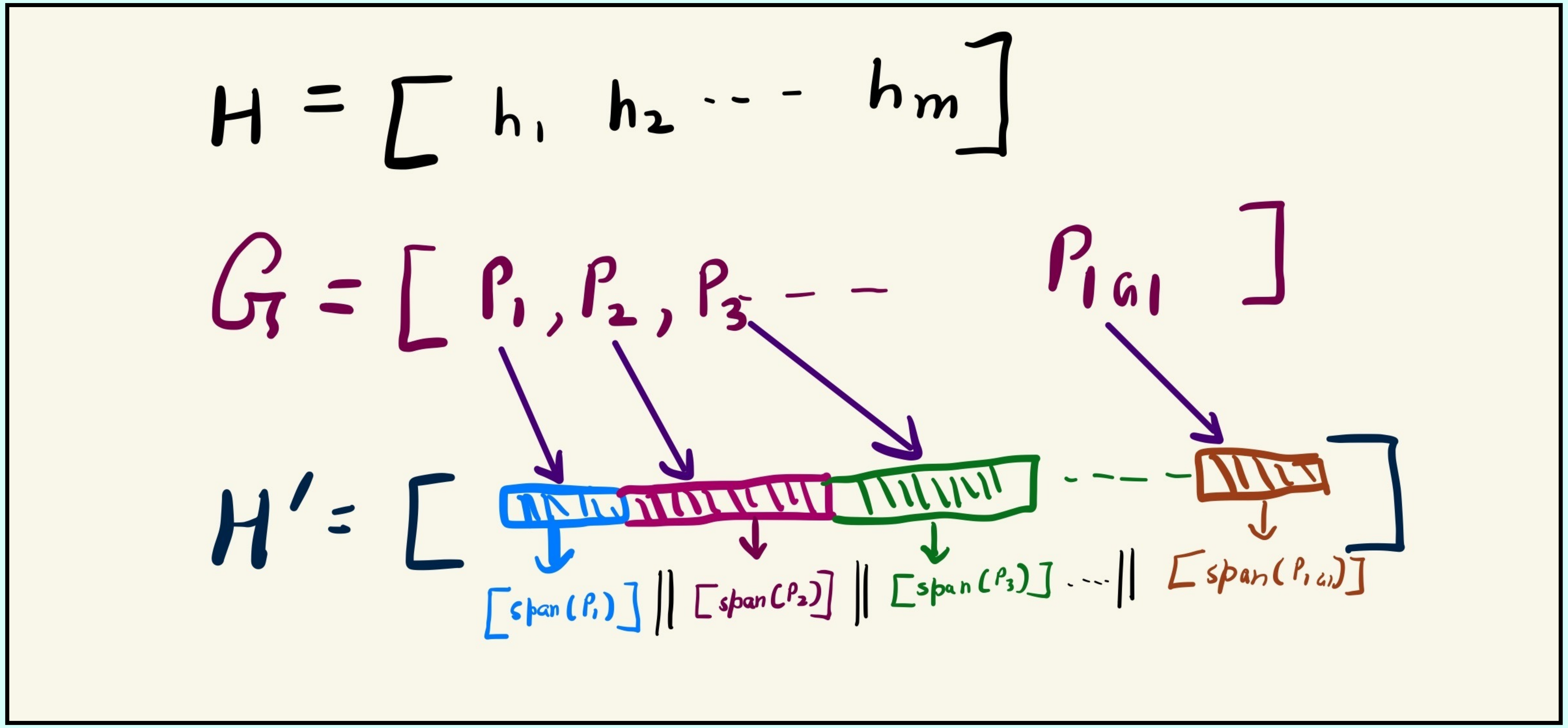

- Take G to be all collection of all subsets of size $\alpha m/k$ of $[m]$.

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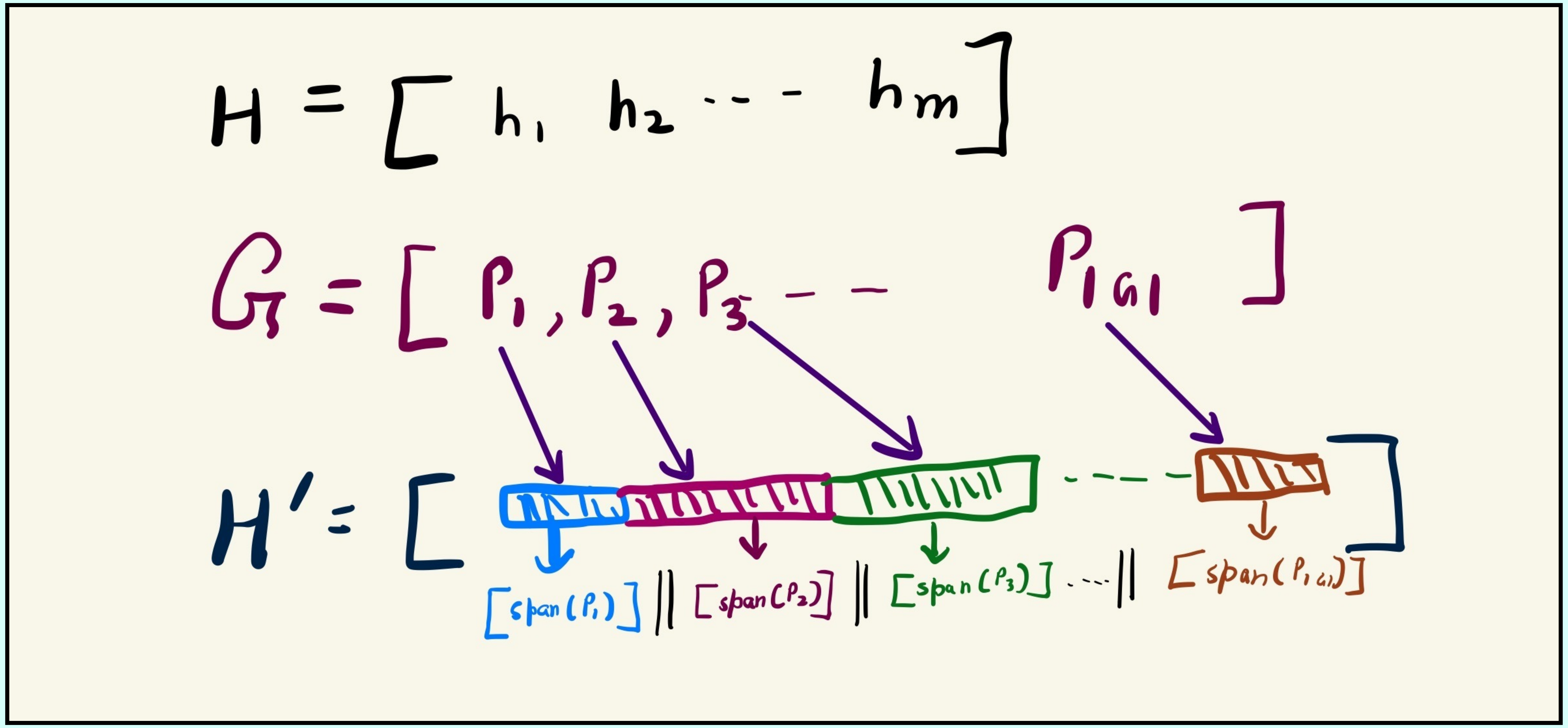

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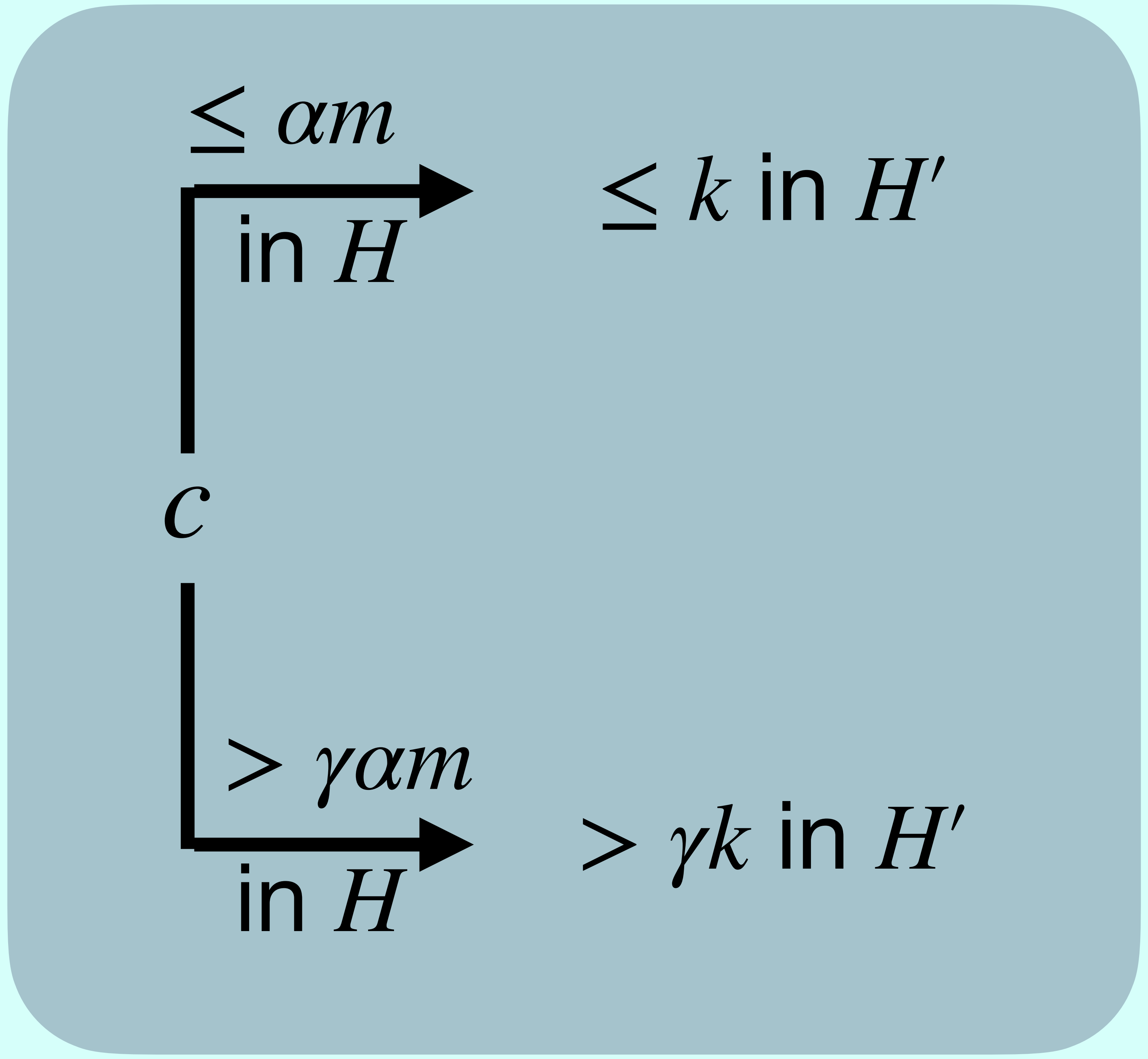

$$
\begin{aligned}
c = {} & \underbrace{x_1 h_1 + \cdots + x_{\alpha m/k} h_{\alpha m/k}}_{\in \operatorname{span}(P_1)} + \\
& \underbrace{x_{\alpha m/k+1} h_{\alpha m/k+1} + \cdots + x_{2\alpha m/k} h_{2\alpha m/k}}_{\in \operatorname{span}(P_2)} + \cdots + \\
& \underbrace{x_{(k-1)\alpha m/k+1} h_{(k-1)\alpha m/k+1} + \cdots + x_{\alpha m} h_{\alpha m}}_{\in \operatorname{span}(P_k)} .
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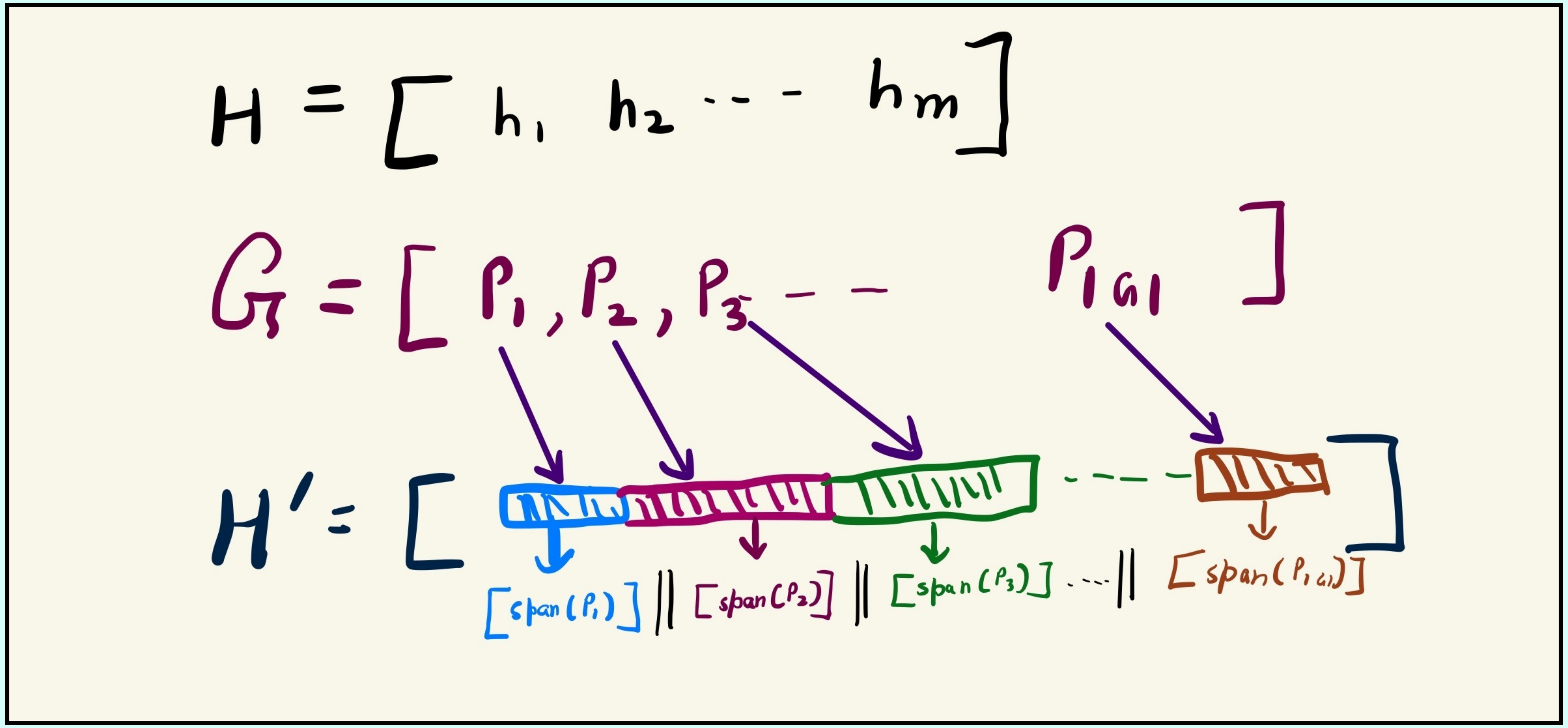

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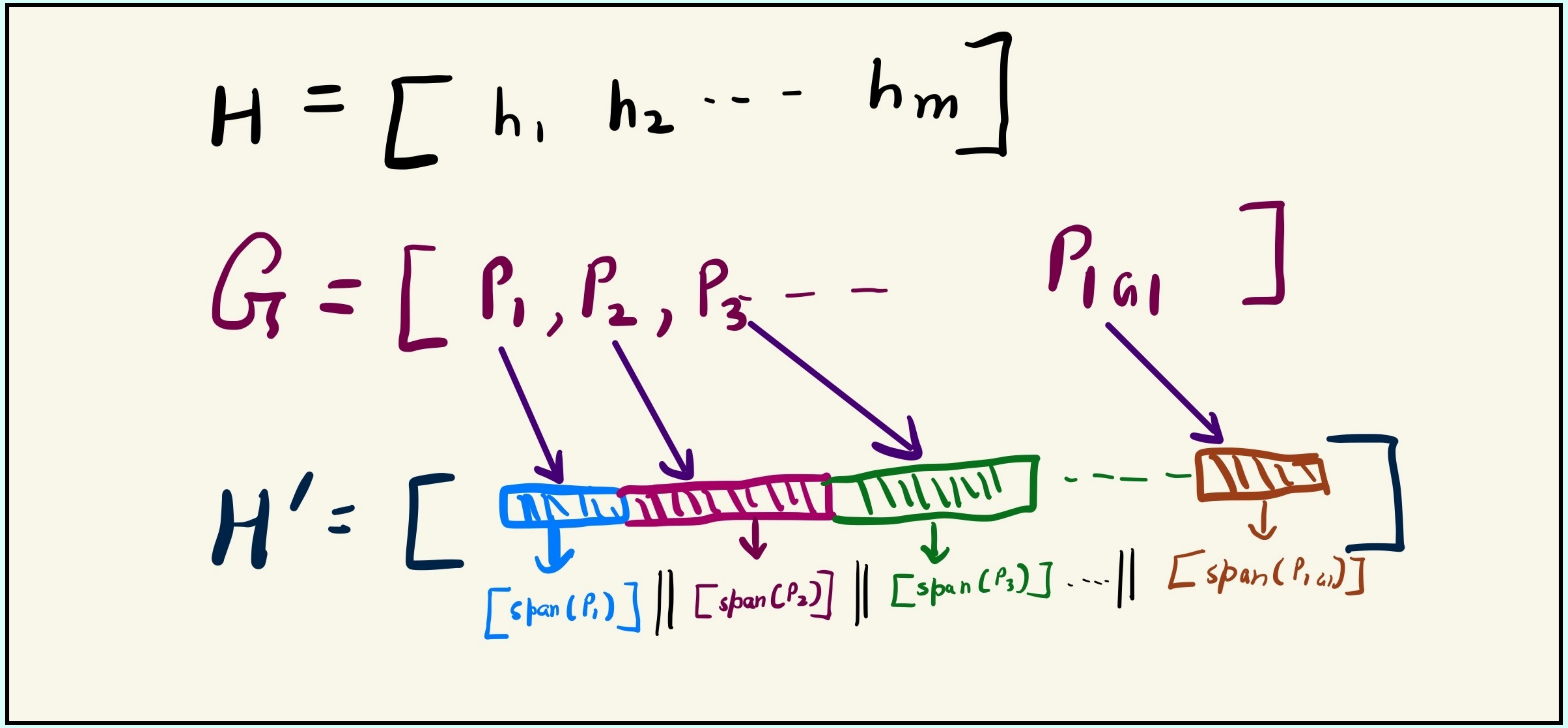

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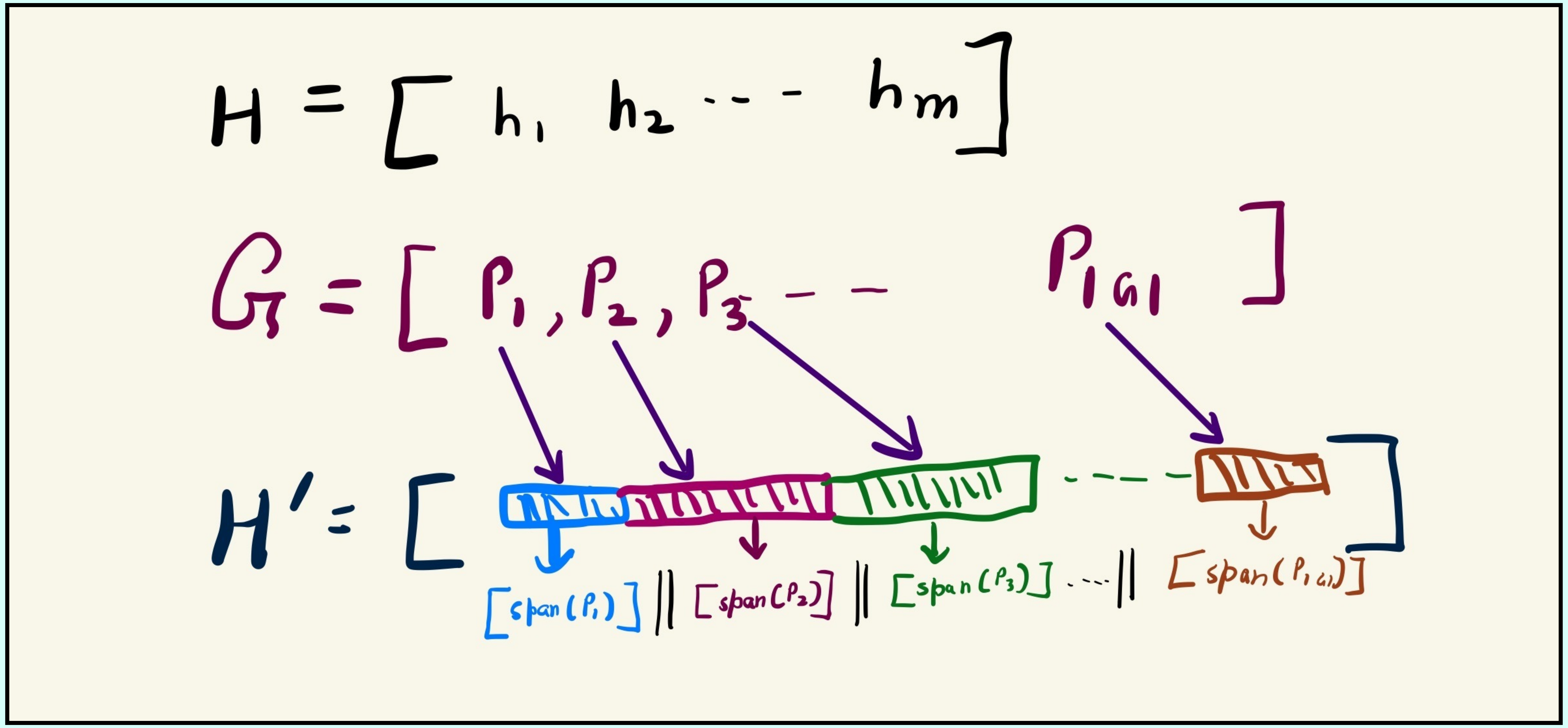

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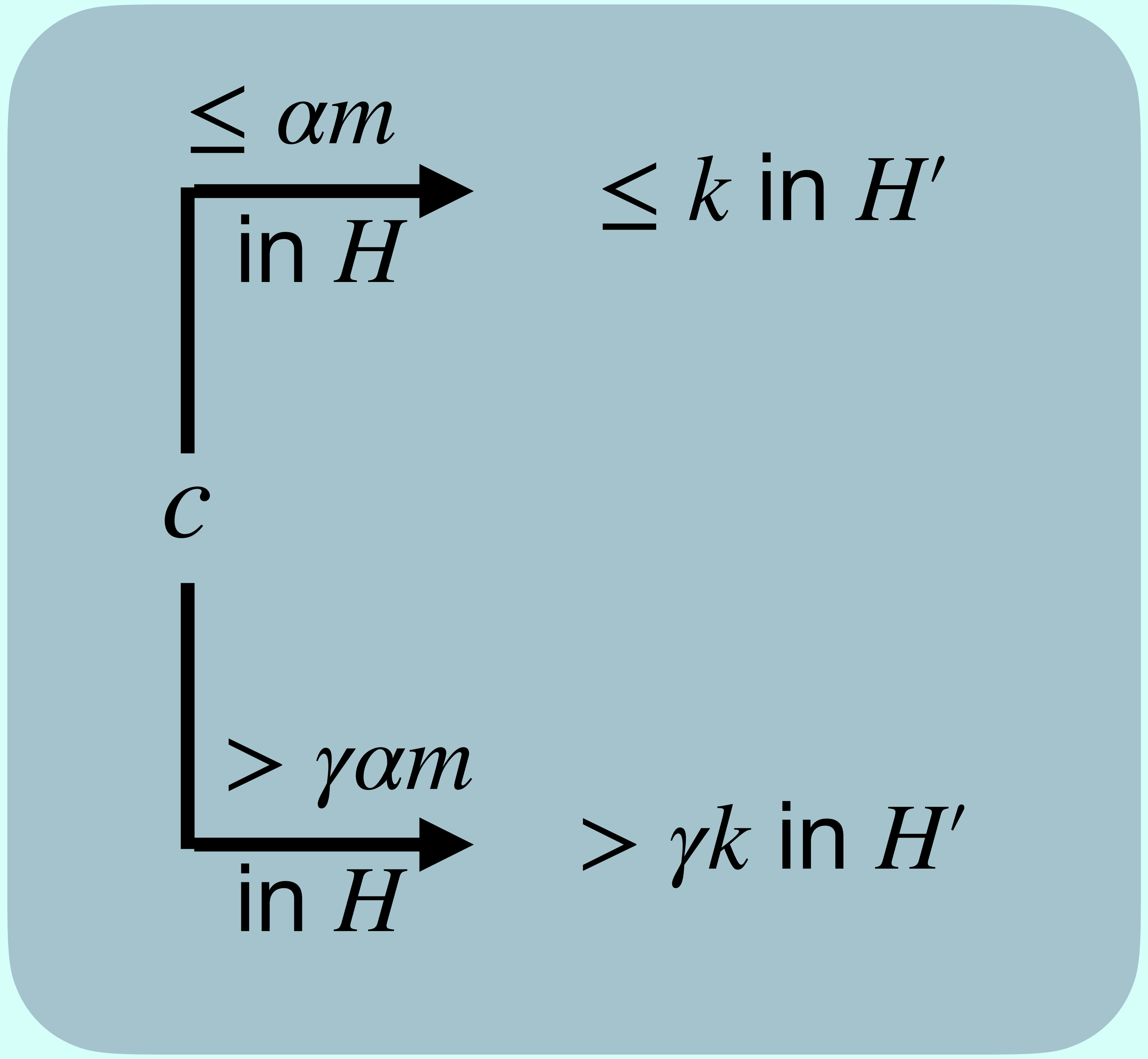

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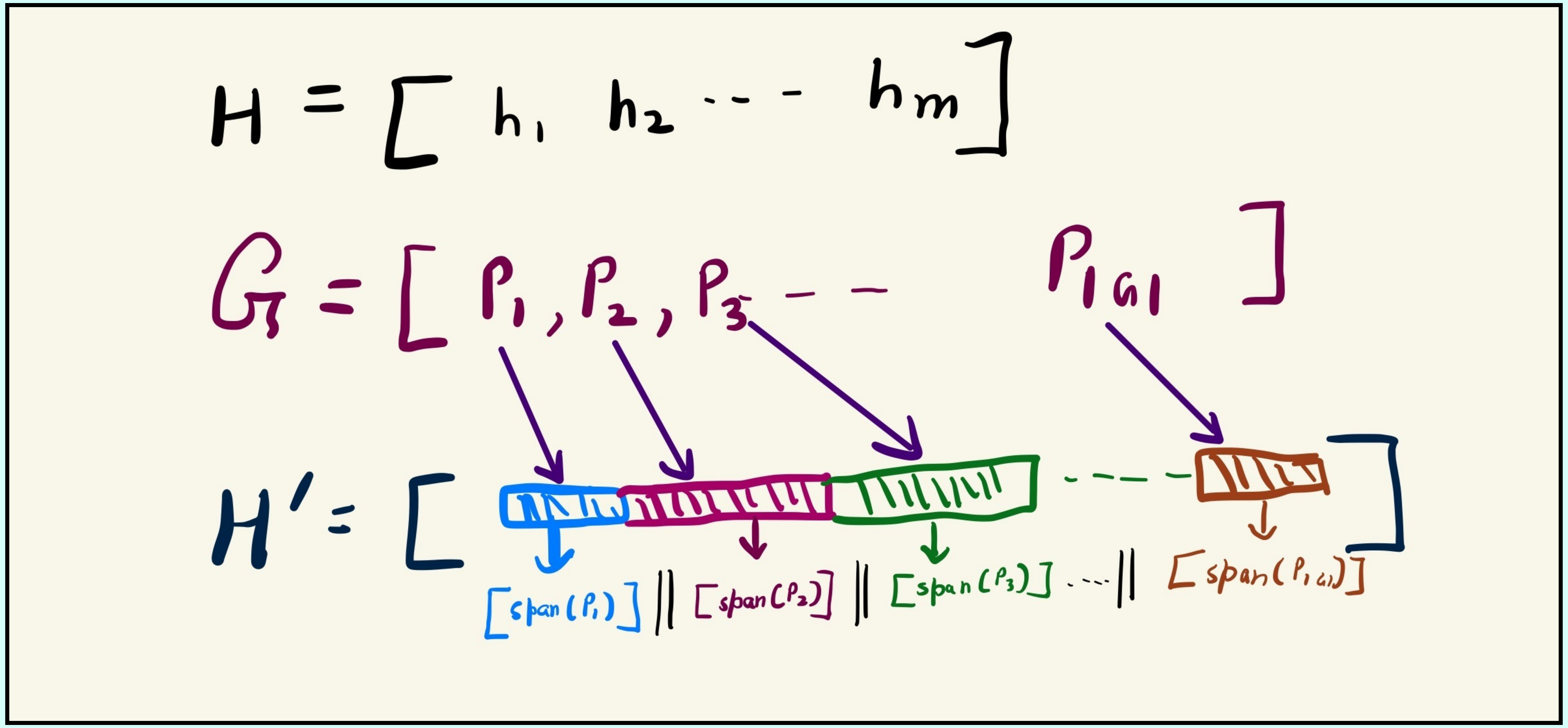

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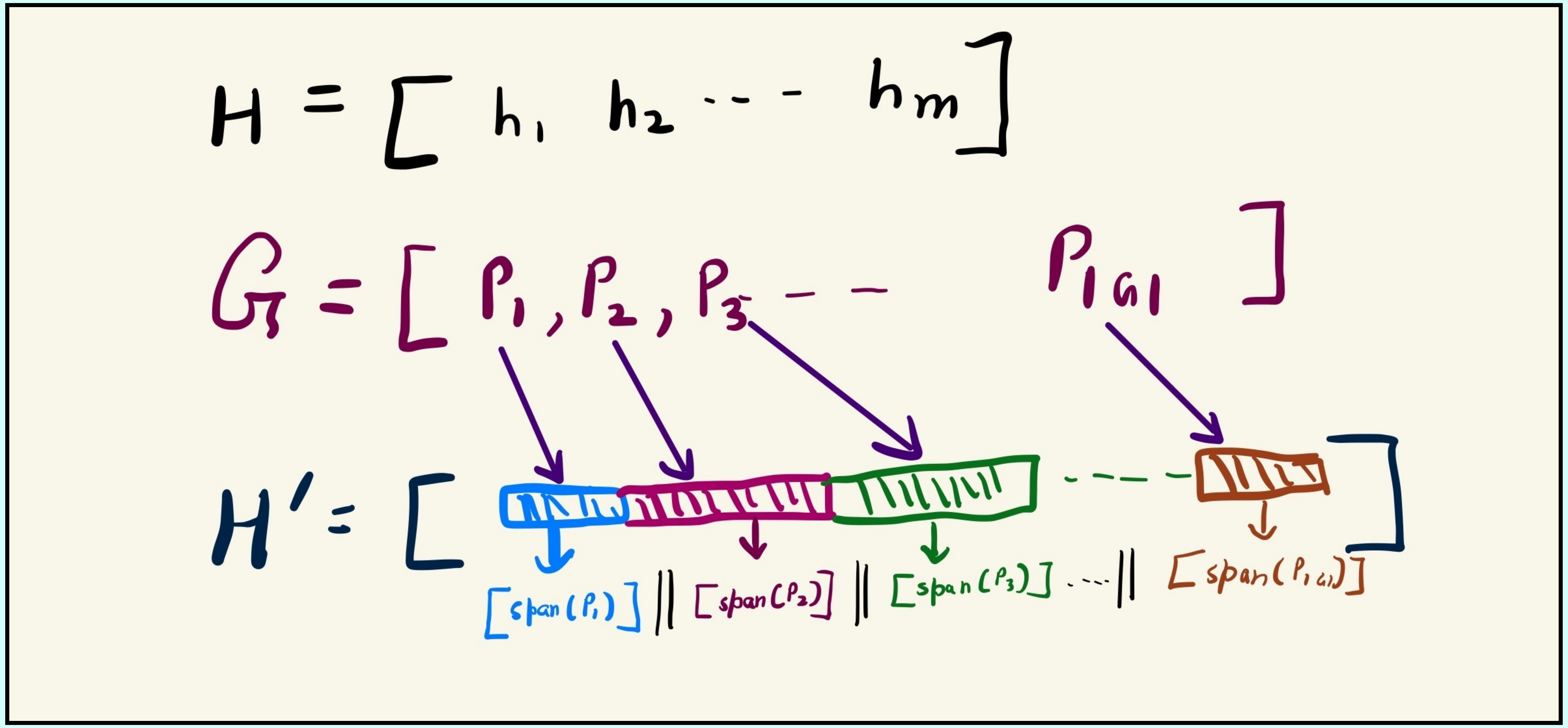

- New size,

$$M = q^{\alpha m/k} \cdot \binom{m}{\frac{\alpha m}{k}}$$

$$\leq 2^{O\left(\frac{m \log k}{k}\right)}$$

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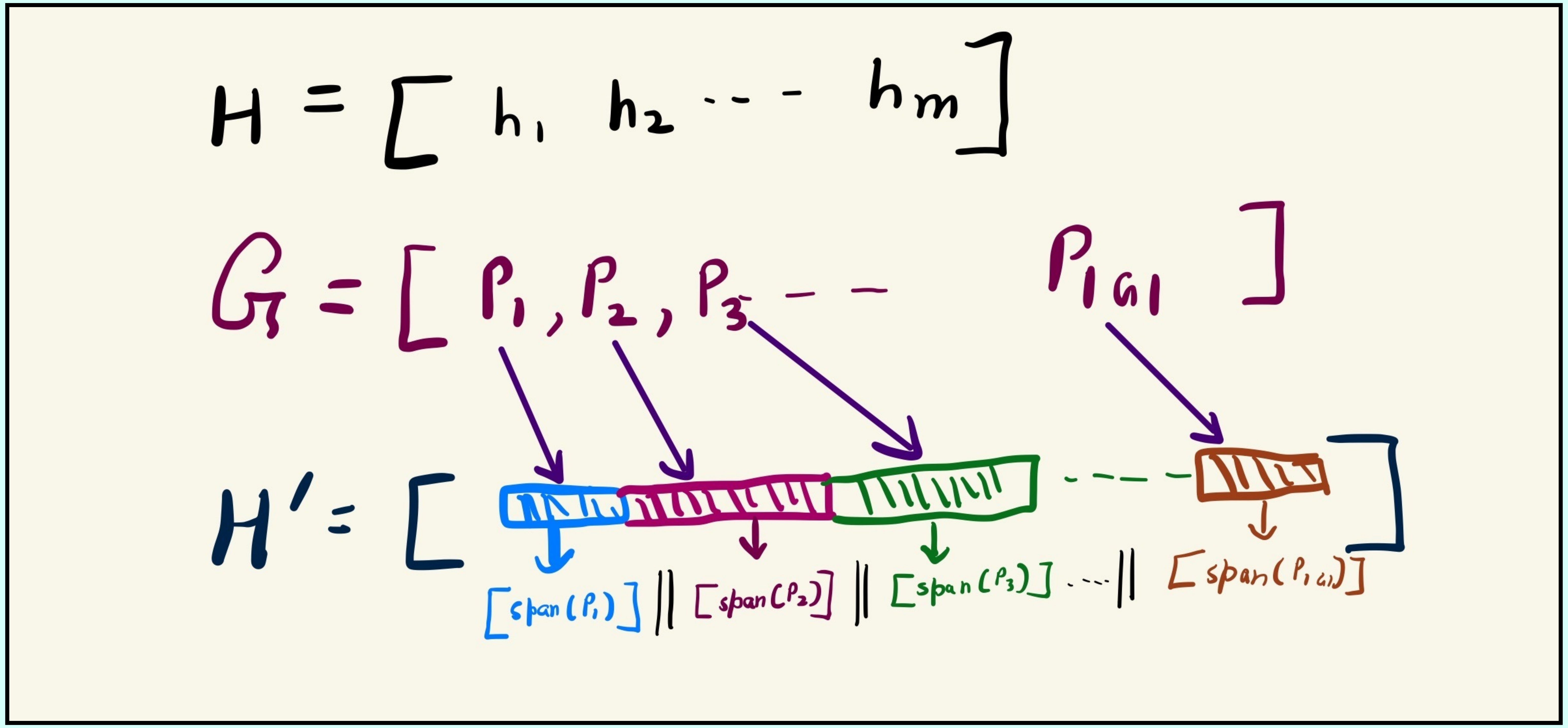

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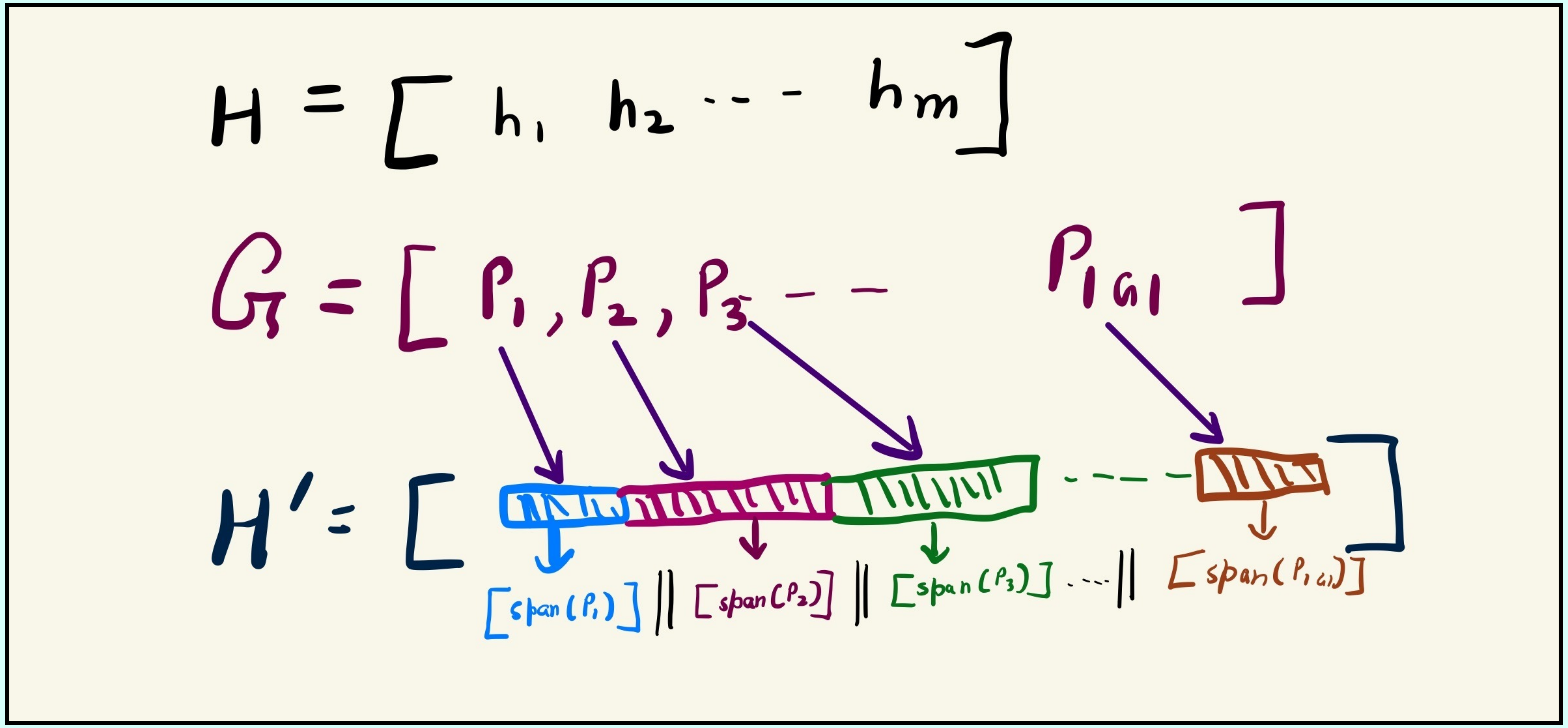

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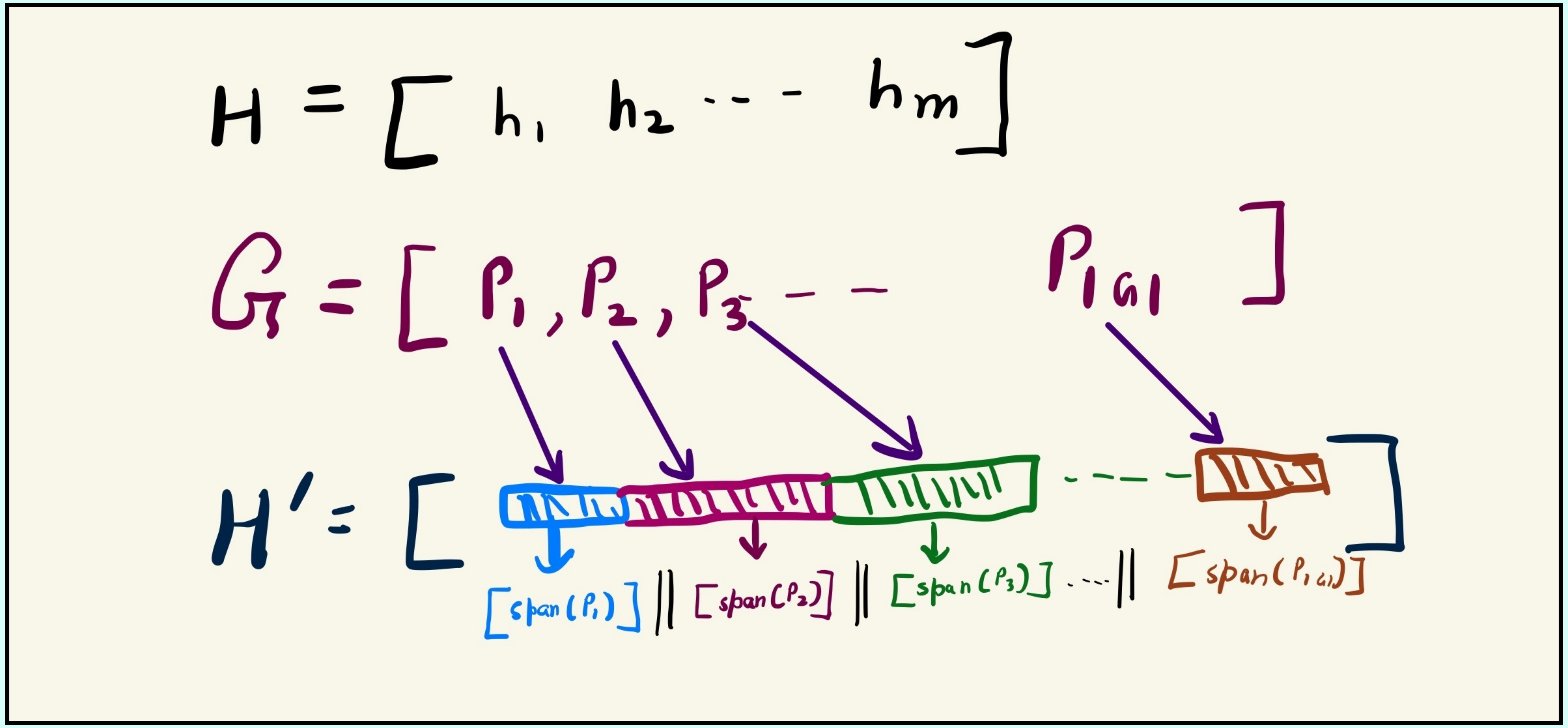

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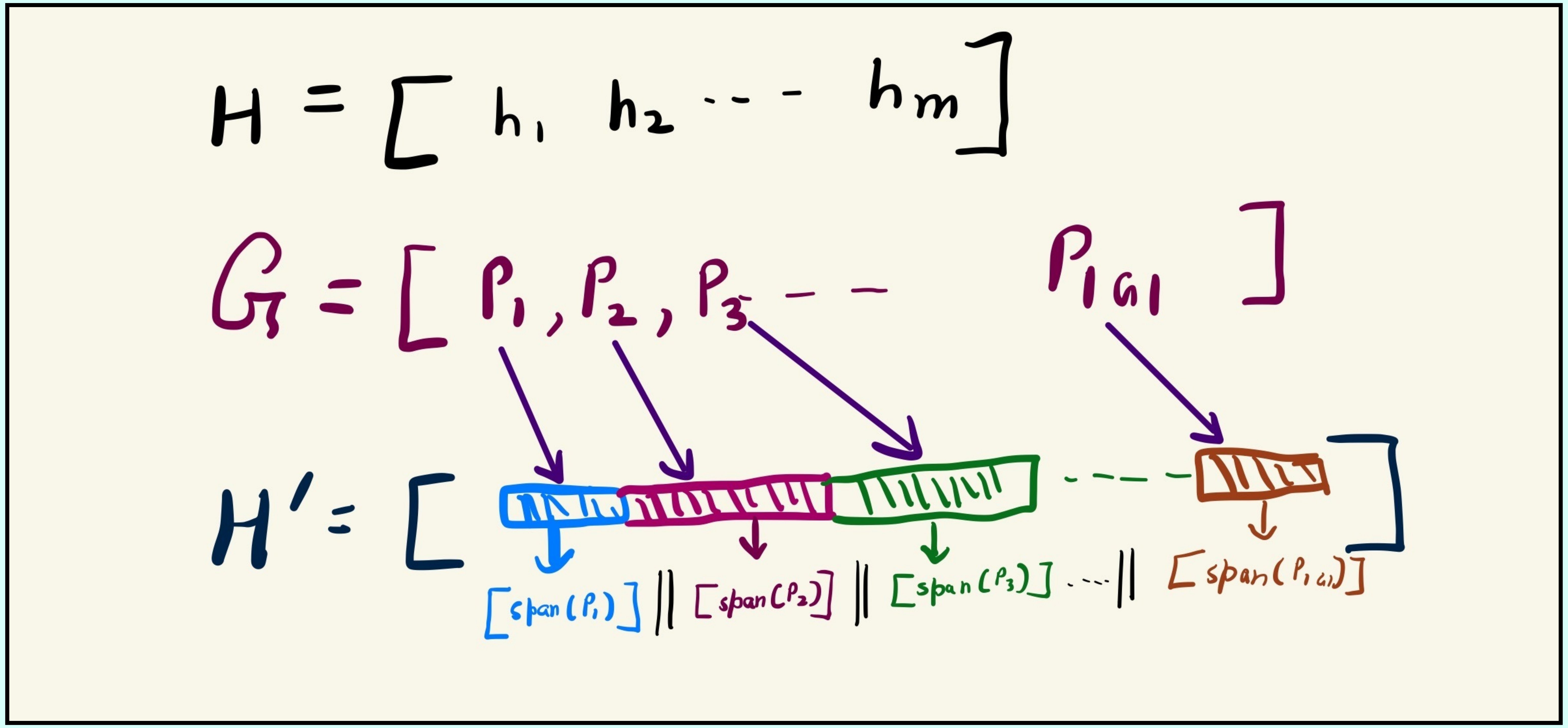

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- Already better !!! 🥳

Cover Familly

Cover Family

$$G = \{ P_1, P_2 \text{ -------- }, P_{|G|} \}$$

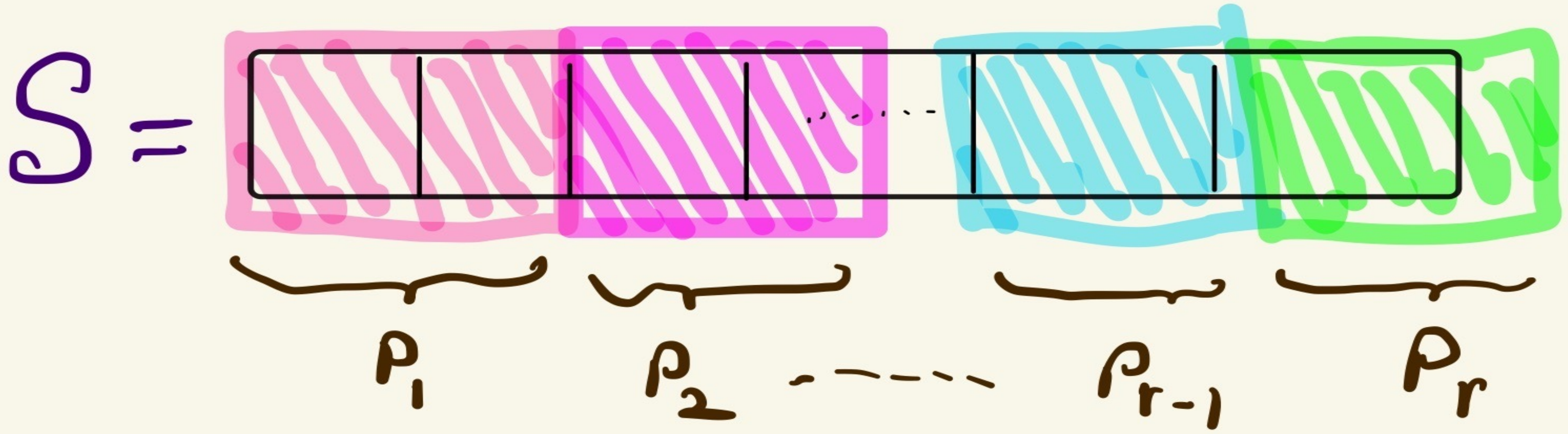
Some collection of subsets of $[m]$

- $P_i \subseteq [m]$
- $|P_i| \leq \frac{(1+\delta)\alpha m}{K}$
- $|G| \leq 2^{O(m/k)}$

We will use these subsets to group cleverly

Cover Family

$|S| = \alpha m$, $S = P_1 \sqcup P_2 \cdots \sqcup P_r$



$r \leq k$.

{Partition Property}

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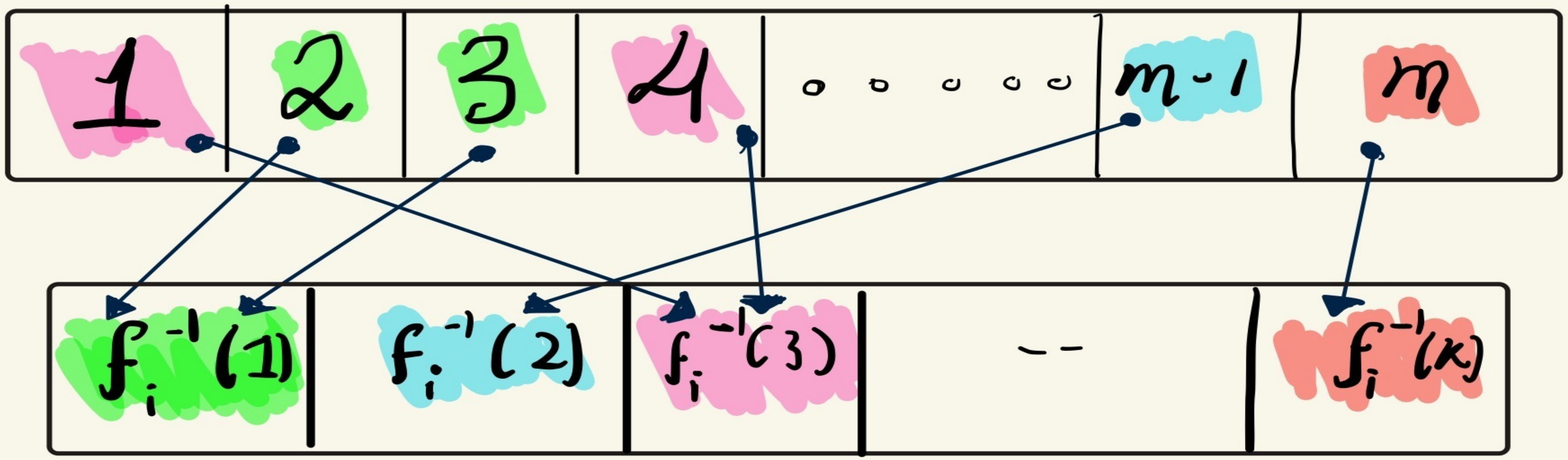
- **Size of Subsets:** Size of all subsets $P \in G$ to be bounded, $|P| \leq (1+\delta)\alpha m/k$.
- **Size of Family:** I also want $|G| = 2^{O(m/k)}$.

Partition Family

Partition Family

$$\mathcal{F} = \{ f_1, f_2, \ldots \ldots f_d \}$$

$$f_i : [m] \longrightarrow [k]$$



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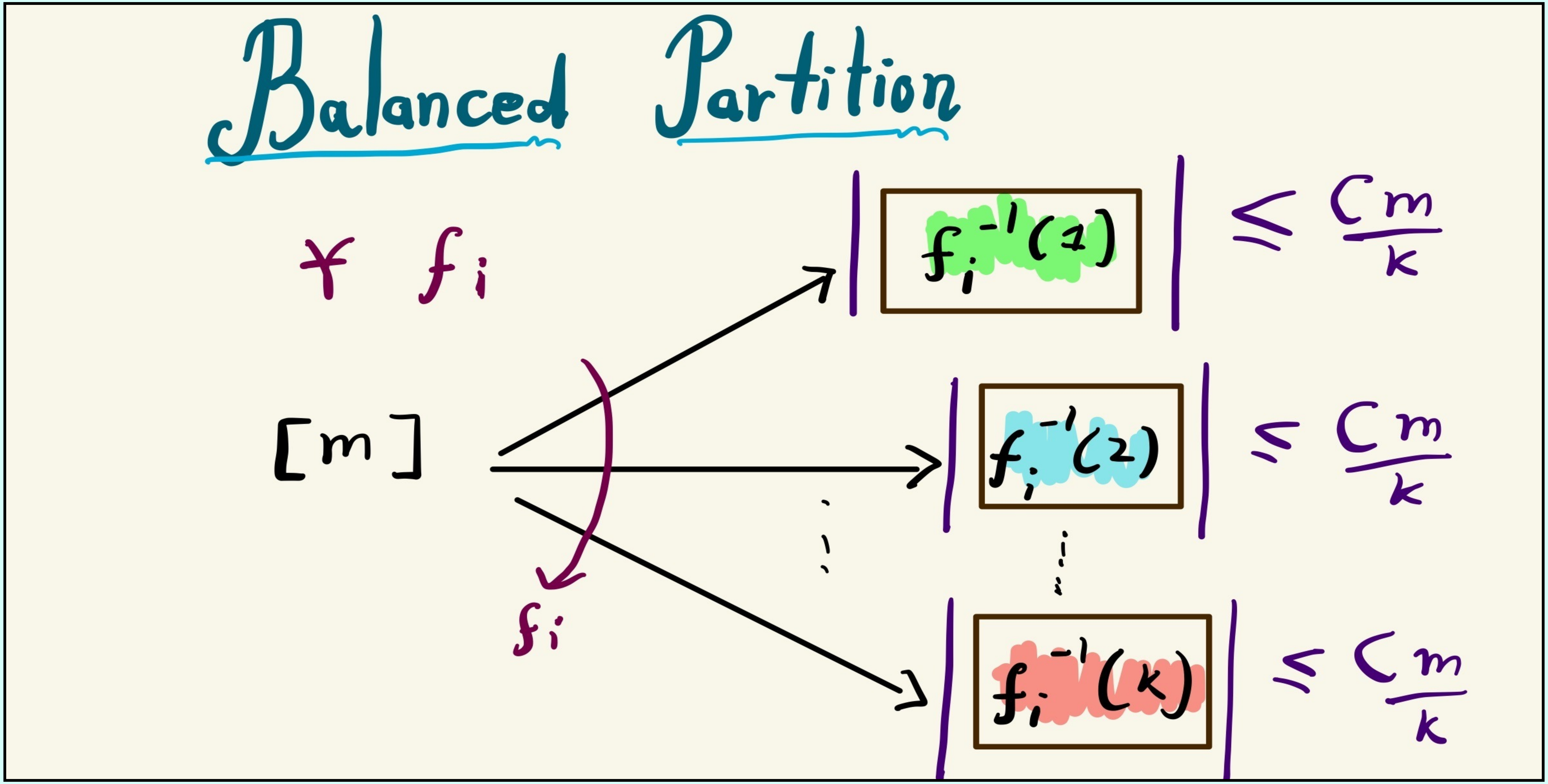
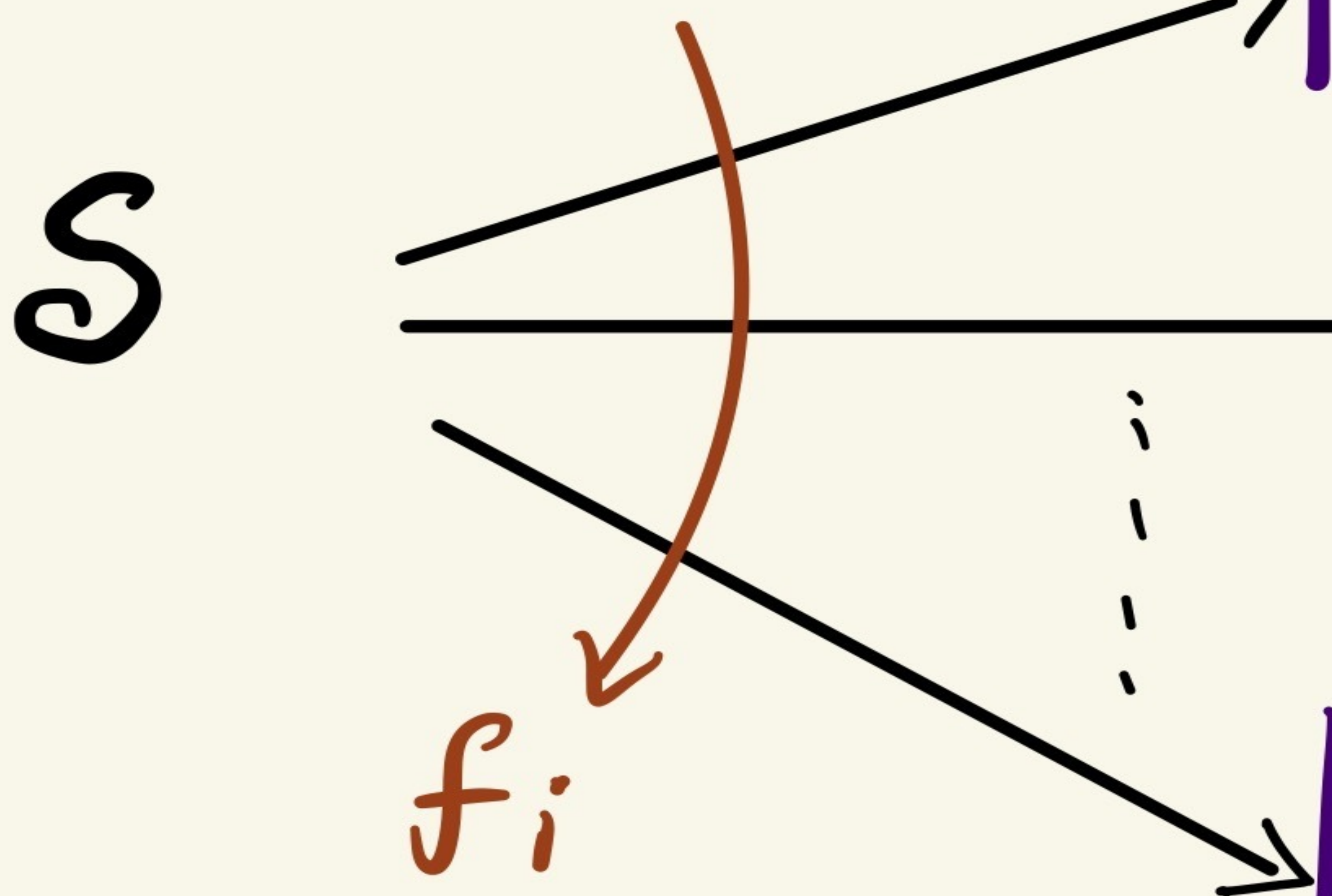


Partition Family

Subset Balancing

$\forall S, |S| = \alpha m, \exists f_i$



$$\left| \boxed{S \cap f_j^{-1}(1)} \right| < \frac{(1+\delta)\alpha m}{k}$$

$$\left| \boxed{S \cap f_j^{-1}(2)} \right| < \frac{(1+\delta)\alpha m}{k}$$

$$\vdots$$

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Partition Family

A set of partition functions $\{f_i\}_{i=1}^{d}$ from $[m]$ to $[k]$ such that,

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- Hence $|G| \leq 2^{Cm/k} \cdot d \cdot k$. (**Size of Family** ✅)
- So we just need to bound d.
- We can show that if we sample f_i's uniformly at random the above can be attained with $d = O(k)$.

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- **Subset Balancing** → **Partition Property** ✅

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Partition Property: For any subset S of αm elements of $[m]$, there exists $\leq k$ disjoint sets in G such that their union is exactly S.

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Partition Property: For any subset S of αm elements of $[m]$, there exists $\leq k$ disjoint sets in G such that their union is exactly S.

- The $\leq k$ disjoint subsets are,

$$\sqcup_{j=1}^{k} \left(f_i^{-1}(j) \cap S\right)$$

- Since, $|f_i^{-1}(j) \cap S| \leq (1+\delta)\alpha m/k$
- This implies $\left(f_i^{-1}(j) \cap S\right) \in G$.

Derandomization

Derandomization of Partitions

$$
\begin{array}{cccccc}
 & 1 & 2 & \cdots & m \\
f_1 \to & g_1(1) & g_2(1) & \cdots & g_m(1) \\
f_2 \to & g_1(2) & g_2(2) & \cdots & g_m(2) \\
f_3 \to & g_1(3) & g_2(3) & \cdots & g_m(3) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
f_d \to & g_1(d) & g_2(d) & \cdots & g_m(d)
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- There will be at least m different functions $\{g_1, g_2 \cdots g_m\}$.
- Write the g_i's as the columns of a table and take partition functions f_i's to be the rows.

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Thank You!