Derandomised tensor product gap amplification for quantum Hamiltonians

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Quantum PCP conjecture

Classical PCP theorem

Given a 3SAT formula ϕ on n bits, it is **NP-hard** to decide whether
(a) ϕ is satisfiable, or
(b) at most a $1 - \delta$ fraction of the clauses of ϕ are satisfiable, for a **constant promise gap** $\delta > 0$.

Quantum PCP conjecture

Local Hamiltonian problem

Input: (classical description of) n-qubit local Hamiltonian $H = \frac{1}{m}\sum_{i=1}^{m} H_i$,
thresholds $\alpha(n), \beta(n)$

Output: Is smallest eigenvalue (ground state energy) of H below $\alpha(n)$ or above $\beta(n)$?

Larger **promise gap** $\beta(n) - \alpha(n)$
→ local Hamiltonian problem becomes easier to solve

Theorem [Kitaev]
The local Hamiltonian problem with **1/poly(n) promise gap** is **QMA-complete**.

Quantum PCP conjecture
The local Hamiltonian problem with **constant promise gap** remains **QMA-complete**.

Dinur's classical PCP proof

Classical PCP theorem

Given a 3SAT formula ϕ on n bits, it is **NP-hard** to decide whether

(a) ϕ is satisfiable, or

(b) at most a $1-\delta$ fraction of the clauses of ϕ are satisfiable, for a **constant promise gap** $\delta > 0$.

3SAT formula on n bits

Deciding satisfiability with $\delta_0 = 1/\mathrm{poly(n)}$ promise gap is NP-hard

↓ **Gap amplification** with parameter t

Constraint satisfaction problem on large alphabet

Deciding satisfiability with $\delta_1 = t \cdot \delta_0$ promise gap is NP-hard

↓ **Alphabet reduction**

3SAT formula on n bits

Deciding satisfiability with $\delta_1 = t \cdot \delta_0$ promise gap is NP-hard

For t constant and $\log n$ iterations:

Gap amplification with parameter t

Quantum gap amplification

Main Theorem (informal)

Transformation on Hamiltonians $H \to H^{(t)}$ such that

- if H acts on n qubits with m terms and locality ℓ, then $H^{(t)}$ acts on $t \cdot n$ qubits with $c(t) \cdot m$ terms and locality $t \cdot \ell$;
- if $\lambda_{min}(H) \leq \alpha$, then $\lambda_{min}\left(H^{(t)}\right) \leq t \cdot \alpha$;
- if $\lambda_{min}(H) \geq \beta$, then $\lambda_{min}\left(H^{(t)}\right) \geq \sqrt{t} \cdot \beta$.

Typical setting:

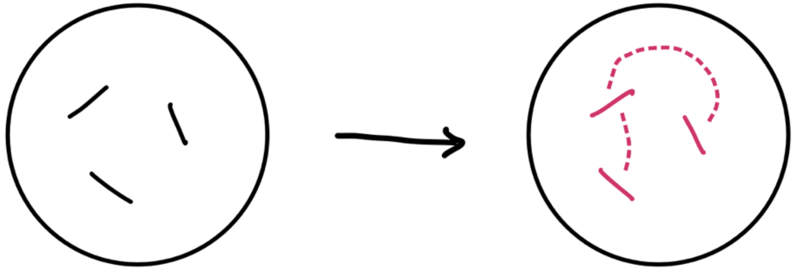
- Start from local Hamiltonian problem with $\alpha = \exp(-n)$ and $\beta = 1/\mathrm{poly}(n)$.
- Set $t = \mathrm{const}$ and repeat $O(\log n)$ times.
- Final Hamiltonian: $\mathrm{poly}(n)$ qubits, terms, and locality; constant promise gap.

Quantum gap amplification

Product amplification

$$H \;\to\; I-(I-H)^t$$

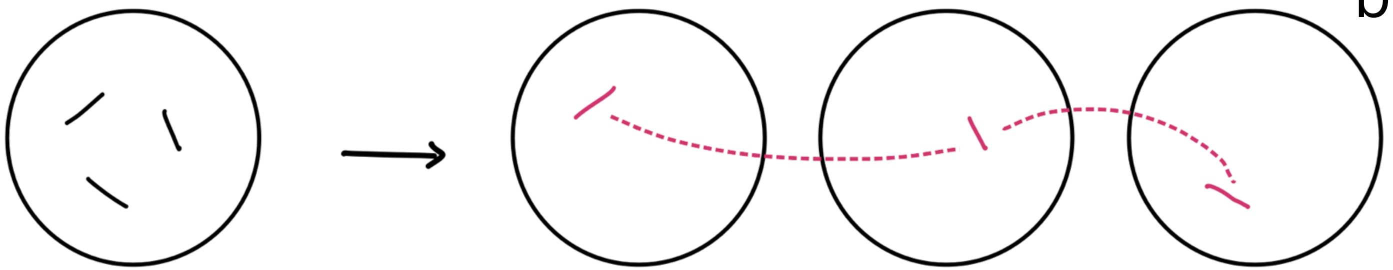
amplifies gap by factor t



Tensor product amplification

$$H \;\to\; I-(I-H)^{\otimes t}$$

amplifies gap by factor t



Quantum gap amplification

Product amplification

$$H \;\to\; I - (I - H)^t$$

$$\mathop{\mathbb{E}}_{i=1}^{m} \Pi_i \;\longrightarrow\; \underbrace{\mathop{\mathbb{E}}_{(i_1,\ldots,i_t)\in[m]^t}}_{m^t \text{ terms}} \left(I - \prod_{j\in[t]} (I - \Pi_{i_j}) \right)$$

Tensor product amplification

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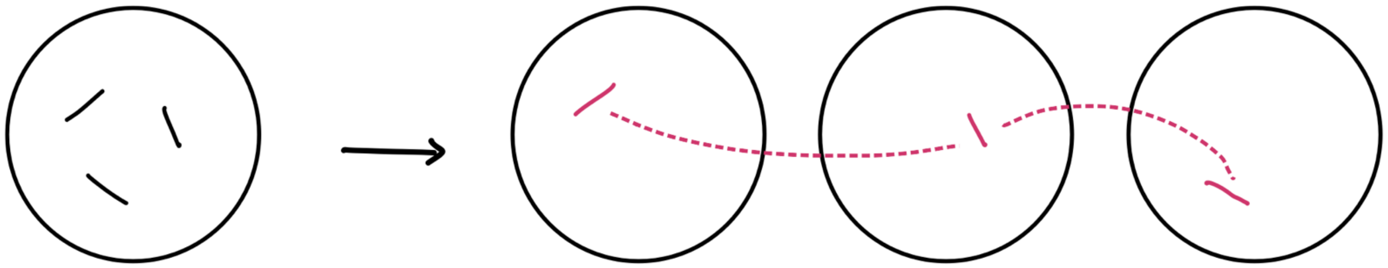

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Quantum gap amplification

Product amplification

$$H \;\to\; I - (I-H)^t$$

[AALV'08]

Key technical tool:
Detectability lemma

$$\mathop{\mathbb{E}}_{i=1}^{m} \Pi_i \;\longrightarrow\; \underbrace{\mathop{\mathbb{E}}_{(i_1,\ldots,i_t)\in\mathcal{F}\subseteq[m]^t}}_{m\cdot D^t\ \text{terms}} \left(I - \prod_{j\in[t]} (I - \Pi_{i_j}) \right)$$

Tensor product amplification

$$H \;\to\; I - (I-H)^{\otimes t}$$

This work

Key technical tool:
Coarse-grained de Finetti

$$\mathop{\mathbb{E}}_{i=1}^{m} \Pi_i \;\longrightarrow\; \underbrace{\mathop{\mathbb{E}}_{(i_1,\ldots,i_t)\in\mathcal{F}\subseteq[m]^t}}_{m\cdot D^t\ \text{terms}} \left(I - \bigotimes_{j\in[t]} (I - \Pi_{i_j}) \right)$$

Quantum gap amplification

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This work

Key technical tool:
Coarse-grained de Finetti

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Main advantage over [AALV'08] product amplification:

- Doesn't require constant locality → can be **iterated with itself**
- New techniques

“Full” tensor product gap amplification

$$H \quad \longrightarrow \quad H^{(t)} = I - (I-H)^{\otimes t} = \mathop{\mathbb{E}}_{(i_1,\ldots,i_t)\in[m]^t} \left(I - \bigotimes_{j\in[t]} (I - \Pi_{i_j}) \right)$$

Ground state of amplified Hamiltonian $H^{(t)}$: $|\psi^{(t)}\rangle = |\psi_{gs}\rangle \otimes \ldots \otimes |\psi_{gs}\rangle$.

If $\langle\psi_{gs}|\, H \,|\psi_{gs}\rangle \geq \beta$, then $\langle\psi^{(t)}|H^{(t)}|\psi^{(t)}\rangle = 1 - (1-\beta)^t \sim t\,\beta$.

”Term-efficient” tensor product gap amplification

$$H \quad \longrightarrow \quad H^{(t)} = \mathop{\mathbb{E}}_{(i_1,\ldots,i_t)\in\mathcal{F}\subseteq[m]^t} \left(I - \bigotimes_{j\in[t]} (I - \Pi_{i_j}) \right)$$

Subsampling of Hamiltonian terms removes tensor product structure

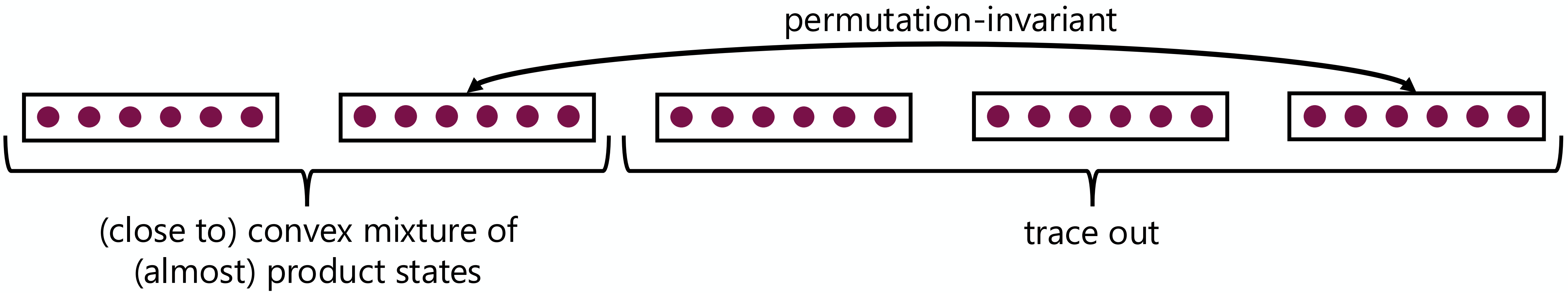
→ ground state $|\tilde{\psi}^{(t)}\rangle$ not necessarily product, **entanglement between copies** could achieve lower energy than tensor product state

Coarse-grained de Finetti technique

(Unattainable) goal: argue that $|\tilde{\psi}^{(t)}\rangle$ is a **product** state.

Can make amplified Hamiltonian **permutation-invariant** (for constant t)
→ ground state $|\tilde{\psi}^{(t)}\rangle$ is **symmetric**

Quantum de Finetti Theorem [Ren'08]
Reduced states of symmetric states are convex combinations of (almost) product states.



Coarse-grained de Finetti technique

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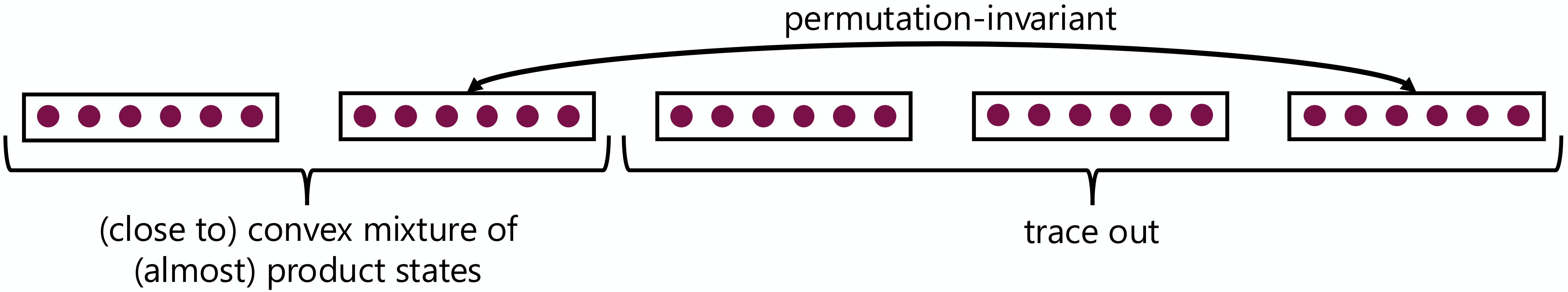

Problem: for good bounds, need to trace out $\mathbf{poly}(\boldsymbol{n})$ systems, but we want to amplify by a **constant** amount t

Coarse-grained de Finetti technique

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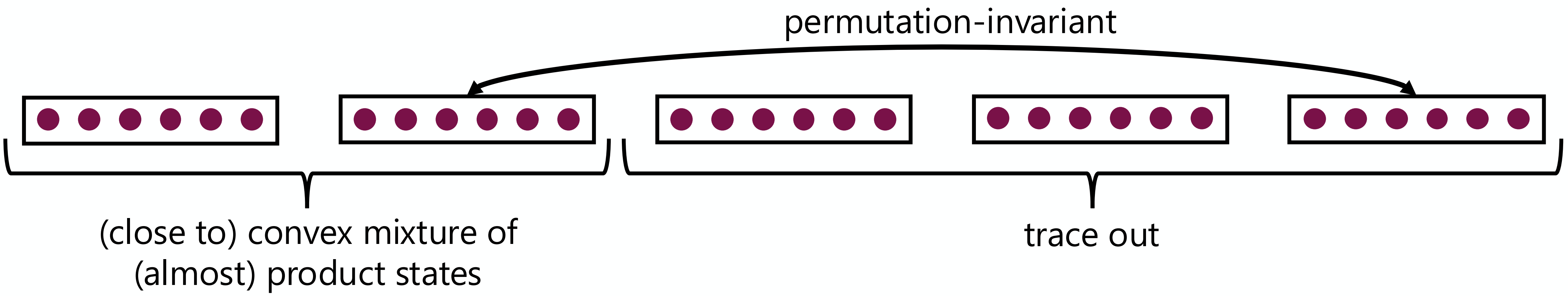

Solution:

Suffices that reduced state **behaves** like product state when **measuring very coarse-grained information**: is the energy w.r.t. to the Hamiltonian terms high or low?

Coarse-grained de Finetti: bounds scale only with number of measurement outcomes (“high” or “low”), not number of qubits in each system

Applications (of sorts) and open questions

Locality-gap tradeoff

QMA-complete family of ℓ-local Hamiltonians with promise gap ϵ
$\rightarrow$ QMA-complete family of ℓ/ϵ-local Hamiltonians with **constant promise gap**

Streaming quantum PCP theorem

QMA-complete family of Hamiltonians with constant promise gap and polynomially many terms, each of which can be measured by looking at only **constantly many qubits at a time**

Quantum games PCP conjecture

Conjecture: QMA can be decided by an MIP* protocol with polylog-length questions and answers, constant completeness-soundness gap, and efficient provers

From gap amplification result: **polylog-length questions**, but poly-length answers

Sparse quantum PCP conjecture

Can we exploit the simple structure of our amplified Hamiltonian to **sparsify** it while retaining the **promise gap**?