Bounds for Hardness Condensation in the Query Model

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Contents

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- Decision Tree Complexity, D(f) := minimum number of queries made by a decision tree algorithm computing f in the worst case.

Prior Work

- [Buresh-Oppenheim and Santhanam'06] first to study the hardness condensation w.r.t circuit-size.

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$f : \{0,1\}^n \to \{0,1\}$ *condenses* to k variables *with respect to* the measure M, if there exists a restriction $\rho\colon [n] \to \{0,1,*\}$ with $|\rho^{-1}(*)| = k$, such that $M(f|_\rho) = \Theta(k)$.

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- **Example:** Degree, Sensitivity.
  ($\deg(f) :=$ degree of the unique multilinear polynomial representing f).

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- [Midrijanis'04] $\mathsf{D}(f) = O(\deg(f)^3$.

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Some Definitions!

- **Block Sensitivity:** $\mathrm{bs}(f, x)$: is the maximum number of disjoint subsets $B_1, B_2, \ldots, B_r$ of $[n]$ such that $f(x) \neq f(x^{B_j})$ for all $j \in [r]$.

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Negative Result

There exists $f : \{0,1\}^n \to \{0,1\}$ such that for every restriction $\rho\colon [n] \to \{0,1,*\}$ with $|\rho^{-1}(*)| = O(\mathrm{bs}(f))$ we have $\mathrm{bs}(f|_\rho) \leq \mathrm{C}(f|_\rho) = O(\mathrm{bs}(f)^{2/3})$.

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Positive Result

Let $f : \{0,1\}^n \to \{0,1\}$ be a Boolean function and $\mathcal{M} \in \{\mathrm{bs}, \mathrm{fbs}, \mathrm{C}\}$ there exists a restriction $\rho \in \{0,1,*\}^n$ with $|\rho^{-1}(*)| = \Theta(\mathcal{M}(f))$ such that $\mathcal{M}(f|_\rho) = \Omega(\sqrt{\mathcal{M}(f)})$.

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Modified Rubinstein function

Define $g\colon \{0,1\}^k \to \{0,1\}$ to be 1 iff the input contains $\sqrt{k}$ consecutive 1's and the rest of the bits are 0's. Then, define $f\colon \{0,1\}^{k^2} \to \{0,1\}$ to be $\mathrm{OR}_k \circ g(x)$, $x \in \{0,1\}^{k^2}$.

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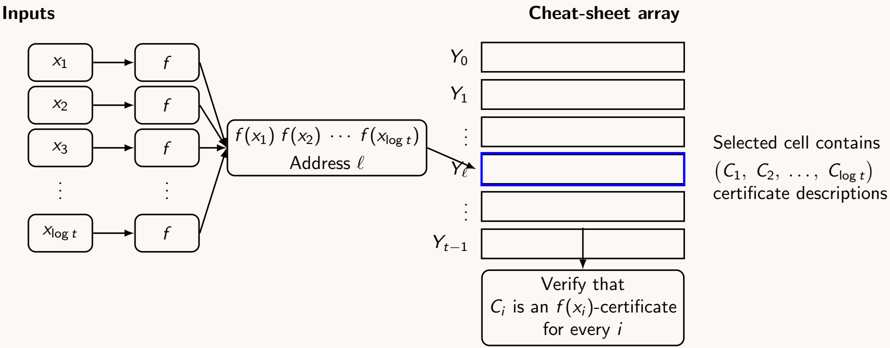
Let's move to Deterministic Query Complexity!

Cheat-sheet construction



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- $f_{\mathsf{CS}}(X) = 1$ iff the cheat-sheet cell Y_ℓ, given by the positive integer ℓ corresponding to the binary string $f(x_1)f(x_2)\cdots f(x_{\log t})$, contains a *description* of $f(x_i)$-certificate for each x_i.

For Deterministic Decision tree complexity

- $\exists$ a function f such that for every restriction $\rho\colon [n] \to \{0,1,*\}$ with $|\rho^{-1}(*)| = O(\mathrm{D}(f))$ we have $\mathrm{D}(f|_\rho) = \tilde{O}(\mathrm{D}(f)^{2/3})$.

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- We will use $\mathrm{Tribe}(x) = (\mathrm{AND}_k \circ \mathrm{OR}_{\sqrt{k}})(x)$ where $x \in \{0, 1\}^{k^{1.5}}$.
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- For every ρ with $|\rho^{-1}(*)| = \tilde{O}(k^{1.5})$, $\mathsf{D}(\mathsf{TribesCS}|\rho) = \tilde{O}(k)$.

Proof (contd.)

Stage 1: Identify a candidate value for $\mathrm{Tribes}(x_i|_\rho)$, $\qquad i \in [\log t]$.
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- Query a popular variable in that OR.
  - If the queried popular variable is 1, then $|O_0|$ decreases by 1.
  - Otherwise, $|V_1|$ shrinks by a factor $\left(1 - \frac{1}{2\sqrt{k}}\right)$.
- Continue until either $|O_0| = O(\sqrt{k} \log k)$ $\quad$ or $\quad |V_1| \leq 2k$.

$$\boxed{\#\text{queries} = O\left(k + \sqrt{k} \log t\right)}$$

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- *Communication complexity:* Does deterministic communication complexity admit a lossless condensation theorem?
- *Other transformations:* Can condensation be established with respect to transformations other than restrictions, e.g., projections?

Thank You!