Quantum–Classical Equivalence for AND Functions

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Joint work with

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When can quantum algorithms achieve
super-polynomial advantage over classical algorithms?

What structure enables a quantum advantage?

What structure rules it out?

Factoring (Shor)

Exponential speedup

Exploits structure (period finding)

Unstructured Search (Grover)

Quadratic speedup only

No exploitable structure

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Is quantum advantage tied to special structure?

Hypothesis

In absence of special structure,
quantum and classical complexities
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How do we test this?

Study concrete computational models.

Lack of structure = f is total

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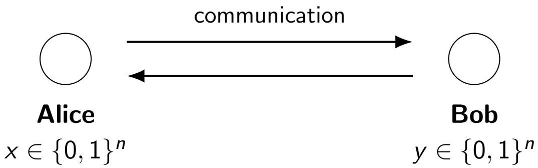
Classical and quantum query complexity
are polynomially equivalent for total functions.

Communication Model



$$F : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$$

Compute $F(x, y)$ with minimal communication.

$$D^{cc}(F), \qquad R^{cc}(F), \qquad Q^{cc}(F)$$

For total functions F,

$$R^{cc}(F) \stackrel{?}{=} (Q^{cc}(F))^{O(1)}$$

Are classical and quantum communication polynomially related?

Wide open.

A Natural Approach

The general communication problem seems out of reach.

Restrict to composed functions

$$F = f \circ g$$

Can we understand quantum vs classical power
in such structured subclasses?

AND-Functions

Let $f : \{0,1\}^n \to \{0,1\}$.

$$F(x,y) = f\big(\mathrm{AND}_2(x_1,y_1),\ldots,\mathrm{AND}_2(x_n,y_n)\big)$$

They capture central problems:

Set-Disjointness

$$\mathrm{DISJ}_n = \mathrm{OR}_n \circ \mathrm{AND}_2$$

Inner Product

$$\mathrm{IP}_n = \mathrm{PARITY}_n \circ \mathrm{AND}_2$$

Prior Progress

Bhurman, de Wolf (2001) Initiated study of AND-functions.

Razborov (2003) For symmetric f:

$$R^{cc}(f \circ \mathrm{AND}_2) = (Q^{cc}(f \circ \mathrm{AND}_2))^2.$$

Open: extend to *all* Boolean f.

Our Result

Main Result. For every total Boolean function f,

$$D^{cc}(f \circ \mathrm{AND}_2) = \left(Q^{cc}(f \circ \mathrm{AND}_2)\right)^7 (\log n)^2.$$

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Corollary. $Q^{cc}(\mathrm{DISJ}_n) = \widetilde{\Omega}(n^{1/7})$

No super-polynomial quantum advantage within this natural class.

More

For $F = f \circ \mathrm{AND}_2$:

Communication measures: $D^{cc}(F),\ R^{cc}(F),\ Q^{cc}(F)$

Matrix measures: $\log \mathrm{rank}(F),\ \log \widetilde{\mathrm{rank}}(F),\ \log \gamma_2(F),\ \log \widetilde{\gamma}_2(F)$

Query / algebraic: $D^{\wedge\text{-}dt}(f),\ \log \mathrm{spar}(f)$

All are polynomially related.

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All are polynomially related.

In particular, the **log approximate-rank conjecture** holds for AND-functions.

Key Tool: Polynomials

Every $f : \{0,1\}^n \to \{0,1\}$ has a unique multilinear polynomial P_f

$$P_f(x) = \sum_{S \subseteq [n]} a_S \prod_{i \in S} x_i.$$

$\deg(f) := \deg(P_f)$ $\qquad\qquad$ $\mathrm{spar}(f) := \#\{\text{monomials in } P_f\}$

$$\widetilde{\deg}(f) = \min_{\substack{p \colon |p(x) - f(x)| \leq 1/3 \\ \forall x}} \deg(p).$$

High Level Proof Idea

Goal:

$$D^{cc}(f \circ \mathrm{AND}_2) \leq \left(Q^{cc}(f \circ \mathrm{AND}_2)\right)^{O(1)}.$$

Characterization of D^{cc} Knop, Lovett, McGuire, Yuan (2021)

$$D^{cc}(f \circ \mathrm{AND}_2) = O\big((\log \mathrm{spar}(f))^5 \log n\big).$$

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Therefore, it suffices to prove

$$(\log \mathrm{spar}(f))^{\Omega(1)} \leq Q^{cc}(f \circ \mathrm{AND}_2).$$

From Quantum Protocols to Rectangles

A quantum protocol using c qubits implies:

$$M_F \approx_{1/3} \sum_i \alpha_i R_i = \sum_i \alpha_i\, g_i(x) h_i(y)$$

where $\sum_i |\alpha_i| \;\leq\; 2^{O(c)}$.

The minimum total weight is, up to constants, is same as **approximate** γ_2 **norm**.

Our main technical result:

$$\log \widetilde{wt}(f \circ \mathrm{AND}_2) \;\geq\; (\log \mathrm{spar}(f))^{\Omega(1)}.$$

Why should large sparsity force large $\widetilde{wt}$?

Rough idea: Restriction Argument (Not Exactly).

From large $\mathrm{spar}(f)$, construct a distribution on restrictions $\rho \sim \mathcal{D}$ such that:

1. With high probability, $\deg\big((f \circ \mathrm{AND}_2)|_\rho\big)$ is large.

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2. Any small-weight rectangle decomposition $\Pi = \sum \alpha_i\, g_i(x) h_i(y)$ for $f \circ \mathrm{AND}_2$ simplifies under ρ, $\deg(\Pi|_\rho)$ is small w.h.p.

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3. Therefore, for some ρ, $(f \circ \mathrm{AND}_2)|_\rho$ has large degree but admits a low-degree approximator — contradicting the fact that for every Boolean q, $\deg(q) \leq O(\widetilde{\deg}(q)^2)$.

What actually works

Restriction + Averaging Argument

From large $\mathrm{spar}(f)$, construct a distribution $\rho \sim \mathcal{D}$ such that:

1. With high probability, $\deg\left(\mathbb{E}_{x_{M(\rho)}}\left[(f \circ \mathrm{AND}_2)|_\rho\right]\right)$ is large.
2. Any small-weight rectangle decomposition $\Pi = \sum_i \alpha_i g_i(x) h_i(y)$ simplifies under restriction and averaging, $\deg\left(\mathbb{E}_{x_{M(\rho)}}[\Pi|_\rho]\right)$ is small w.h.p.
3. Contradicting the degree vs. approximate-degree relation.

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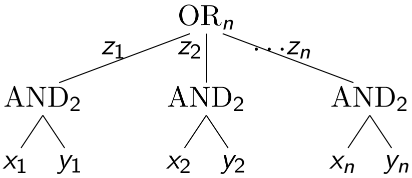
3. Contradicting the degree vs. approximate-degree relation.

Conclusion: Large sparsity $\Rightarrow$ large $\widetilde{wt}$.

Concrete Example: Set-Disjointness

$$\neg\mathrm{DISJ}_n = \mathrm{OR}_n \circ \mathrm{AND}_2$$



High-Level Idea: Restriction + Averaging

1. For a suitable random restriction $\rho \sim \mathcal{D}$, the function $\mathbb{E}_{x_{M(\rho)}}[(\mathrm{OR}_n \circ \mathrm{AND}_2) \mid_\rho]$ has large degree with high probability.
2. Any decomposition $\Pi = \sum_i b_i\, g_i(x) h_i(y) \approx_\varepsilon M_F$ with small $\sum_i |b_i|$ simplifies under restriction and averaging.

Properties of OR_n

- For every restriction $\rho_z \in \{0, \star\}^n$,

$$\deg(\mathrm{OR}_n \mid_{\rho_z}) = |\rho_z^{-1}(\star)|.$$

  $\mathrm{OR}_n \mid_{\rho_z}$ computes OR on the free variables.
- This is exactly what gives OR_n large sparsity.
- We lift these restrictions naturally to restrictions for $\mathrm{OR}_n \circ \mathrm{AND}_2$.

Lifting Restrictions from OR_n to $\mathrm{OR}_n \circ \mathrm{AND}_2$

Given a restriction $\rho_z \in \{0, \star\}^n$, define a lifted restriction

$$\rho := \mathrm{Lift}(\rho_z).$$

For every coordinate $i \in [n]$,

$$(\rho(x_i), \rho(y_i)) = \begin{cases} (\Delta, 0), & \rho_z(z_i) = 0, \\ (\star, 1), & \rho_z(z_i) = \star. \end{cases}$$

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- ρ induces ρ_z on the z-variables: $(x_i \wedge y_i)\,|_\rho = \rho_z(z_i)$.
- Define the masked variables $M(\rho) := \{x_i : \rho(x_i) = \Delta\}$.
  The masked variables remain free, but $(\mathrm{OR}_n \circ \mathrm{AND}_2)\,|_\rho$ is independent of them.
- Hence $(\mathrm{OR}_n \circ \mathrm{AND}_2)\,|_\rho$ computes OR on the $\star$-variables, and therefore

$$\deg\left(\mathbb{E}_{x_{M(\rho)}}\left[(\mathrm{OR}_n \circ \mathrm{AND}_2)\,|_\rho\right]\right) = |\rho^{-1}(\star)|.$$

Restriction-and-Averaging Process

- Sample a random restriction
$$\rho_z \in \{0, \star\}^n$$
having exactly pn stars uniformly at random, where $p = \Theta(n^{-1/2})$.
- Lift ρ_z to a restriction
$$\rho := \mathrm{Lift}(\rho_z).$$
Let $\mathcal{D}$ denote the resulting distribution over lifted restrictions.
- Apply the restriction-and-averaging operator:
$$\mathbb{E}_{x_{M(\rho)}}[\,\cdot\,], \qquad \rho \sim \mathcal{D}.$$

$\mathrm{OR}_n \circ \mathrm{AND}_2$ Retains Hardness

For every $\rho \sim \mathcal{D}$,

$$\deg\left(\mathbb{E}_{x_{M(\rho)}}\left[(\mathrm{OR}_n \circ \mathrm{AND}_2)\mid_\rho\ \right]\right) = |\rho^{-1}(\star)| = pn = \Theta(n^{1/2}).$$

Simplification of Small-Weight Approximator for $\mathrm{OR}_n \circ \mathrm{AND}_2$

Let $\Pi = \sum_i b_i g_i(x) h_i(y)$ approximate $\mathrm{OR}_n \circ \mathrm{AND}_2$ and suppose $\sum_i |b_i| \le 2^{O(n^{1/4})}$.

Observations / Claims

1. All y-variables are fixed under every $\rho \sim \mathcal{D}$.
2. Under ρ, each rectangle collapses to a function only of the x-variables.
3. (Fourier tail decay property) For every Boolean function $g : \{0,1\}^n \to \{0,1\}$, the expected Fourier mass above level k decays as $2^{-\Omega(k)}$ under the restriction-and-averaging procedure. For $G_\rho := \mathbb{E}_{x_{M(\rho)}}[g|_\rho]$,

$$\mathbb{E}_{\rho\sim\mathcal{D}}\big[\|\widehat{G}_\rho\|_1^{\ge k}\big] \le 2^{-\Omega(k)}.$$

4. Using the tail bound with $k = n^{1/4}$, there is a restriction ρ such that

$$\left\|\widehat{\mathbb{E}_{x_{M(\rho)}}[\Pi|_\rho]}\right\|_1^{\ge k} \le \frac{1}{10}.$$

Putting Things Together

- For every $\rho \in \mathcal{D}$,

$$\deg\Big(\mathbb{E}_{x_{M(\rho)}}\left[(\mathrm{OR}_n \circ \mathrm{AND}_2)\mid_\rho\right]\Big) = \Theta(\sqrt{n}).$$

- Let Π be a $1/3$-approximator for $\mathrm{OR}_n \circ \mathrm{AND}_2$ with $\sum_i |b_i| \leq 2^{O(n^{1/4})}$.
- We saw that there exists $\rho \in \mathcal{D}$ such that, after restriction-and-averaging, the Fourier tail above level $k = n^{1/4}$ has ℓ_1-mass at most $1/10$.
- Truncate above degree k. This gives a degree $n^{1/4}$ approximator to a Boolean function of degree $\Theta(\sqrt{n})$, contradicting

$$\widetilde{\deg}(f) \geq c\sqrt{\deg(f)}.$$

Thus $\sum_i |b_i| \geq 2^{\Omega(n^{1/4})}$.

Step Back: What Structure Did We Use?

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The same is true for parity:

$$\deg(\mathrm{PARITY}_n|_{\rho_z}) = |\rho_z^{-1}(\star)|.$$

Therefore,

$$\widetilde{wt}(\mathrm{PARITY}_n \circ \mathrm{AND}_2) = 2^{\Omega(n^{1/4})}.$$

Towards General Functions with Large Sparsity

Can we always find such a restriction structure whenever sparsity is large?

Perhaps after fixing some variables?

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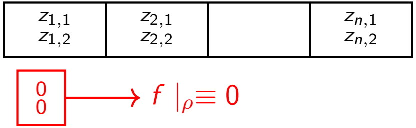
$$f := \mathrm{AND}_n \circ \mathrm{OR}_2, \qquad \mathrm{spar}(f) = 3^n.$$



Structure of Non-Sparse Functions

Suppose f has large sparsity.

Then we can find a set of *core variables* $V \subseteq \{x_1, \ldots, x_n\}$ such that:

$$\forall \alpha \in \{0, \star\}^V, \quad \exists\, \beta(\alpha) \in \{0,1\}^{[n] \setminus V}$$

for which the restriction $\rho_z := \alpha \cup \beta(\alpha)$ satisfies $\deg(f\,|_{\rho_z}) = |\rho_z^{-1}(\star)|$.

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for which the restriction $\rho_z := \alpha \cup \beta(\alpha)$ satisfies $\deg(f\mid_{\rho_z}) = |\rho_z^{-1}(\star)|$.

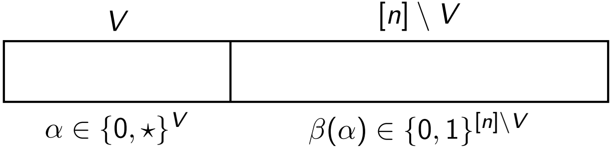
Define the family of restrictions

$$\mathcal{R} := \left\{ \alpha \cup \beta(\alpha) : \alpha \in \{0, \star\}^V \right\}.$$

We will call them *semi-adaptive max-degree restriction trees*.

Structure of Semi-Adaptive Restrictions

A restriction in $\mathcal{R}$ has the following form:



$$\rho_z = \alpha \cup \beta(\alpha), \qquad \deg(f\mid_{\rho_z}) = |\rho_z^{-1}(\star)|.$$

The restriction process is adaptive only on the non-core variables.

Lifting Restrictions to $f \circ \mathrm{AND}_2$

Given $\rho_z \in \mathcal{R}$, define a lifted restriction $\rho = \mathrm{Lift}(\rho_z)$ on the variables (x_i, y_i).

For each coordinate i,

$$(\rho(x_i), \rho(y_i)) = \begin{cases} (\Delta, 0), & \rho_z(z_i) = 0 \text{ and } i \in V, \\ (\star, 1), & \rho_z(z_i) = \star \text{ and } i \in V, \\ (1, 1), & \rho_z(z_i) = 1 \text{ and } i \notin V, \\ (1, 0), & \rho_z(z_i) = 0 \text{ and } i \notin V. \end{cases}$$

Then

$$(x_i \wedge y_i) \mid_\rho = \rho_z(z_i).$$

Properties of Lifted Restrictions

Let $\rho = \mathrm{Lift}(\rho_z)$.

1. The lifted restriction induces ρ_z on the z-variables. Hence
$$\deg((f \circ \mathrm{AND}_2)\restriction_{\rho}) = \deg(f\restriction_{\rho_z}) = |\rho_z^{-1}(\star)| = |\rho^{-1}(\star)|.$$

2. $(f \circ \mathrm{AND}_2)\restriction_{\rho}$ is independent of the masked variables. Therefore,
$$\deg\Big(\mathbb{E}_{x_{M(\rho)}}\big[(f \circ \mathrm{AND}_2)\restriction_{\rho}\big]\Big) = |\rho^{-1}(\star)|.$$

3. All y-variables are fixed, and every non-core x-variable is fixed to 1.

General Functions with Large Sparsity

The lifted restrictions preserve hardness under restriction-and-averaging:

$$\deg\Big(\mathbb{E}_{x_{M(\rho)}}\big[(f \circ \mathrm{AND}_2)\mid_\rho\big]\Big) = |\rho^{-1}(\star)|.$$

Therefore, the previous lower-bound framework applies to $f \circ \mathrm{AND}_2$.

The simplification of small-weight decompositions still works.

- Any function on the x-variables has exponential Fourier tail decay under restriction-and-averaging.
- The only change in the analysis is that the ambient dimension n is replaced by the number of core variables $|V|$.

Semi-Adaptive Max-Degree Restriction Trees

Theorem

$f : \{0,1\}^n \to \{0,1\}$ *admits a semi-adaptive max-degree restriction tree of depth*

$$\Omega\left(\frac{\log \mathrm{spar}(f)}{\log n}\right).$$

Theorem

For every $f : \{0,1\}^n \to \{0,1\}$,

$$\log \widetilde{wt}(f \circ \mathrm{AND}_2) = \Omega\left(\left(\frac{\log \mathrm{spar}(f)}{\log n}\right)^{1/4}\right).$$

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$$\log \widetilde{wt}(f \circ \mathrm{AND}_2) = \Omega\left(\left(\frac{\log \mathrm{spar}(f)}{\log n}\right)^{1/4}\right).$$

$$D^{cc}(f \circ \mathrm{AND}_2) = O(D^{\wedge\text{-dt}}(f)) = O\big((\log \mathrm{spar}(f))^{O(1)} \log n\big)$$

$$\log \widetilde{wt}(f \circ \mathrm{AND}_2) = O(Q^{cc}(f \circ \mathrm{AND}_2).$$

Constructing Semi-Adaptive Restriction Trees

Theorem

Let f have sparsity s. Then f admits a semi-adaptive max-degree restriction tree of depth $\Omega\left(\frac{\log s}{\log n}\right)$.

Proof Idea. Write

$$f(x) = \sum_{S \subseteq [n]} a_S \prod_{i \in S} x_i, \qquad \mathcal{M}_f := \{S : a_S \neq 0\}.$$

Since $|\mathcal{M}_f| = s$, Sauer–Shelah gives a shattered set

$$V \subseteq [n], \qquad |V| = \Omega\left(\frac{\log s}{\log n}\right).$$

Shattering means: for every $A \subseteq V$, some monomial has intersection with V exactly A.

Thus, after choosing the free core variables, we can fix non-core variables so that a full-degree monomial on the free variables survives.

Our Results

For $F = f \circ \mathrm{AND}_2$:

$$D^{cc}(F),\ R^{cc}(F),\ Q^{cc}(F)$$

$$\log \mathrm{rank}(F),\ \log \widetilde{\mathrm{rank}}(F),\ \log \gamma_2(F),\ \log \widetilde{\gamma}_2(F)$$

$$D^{\wedge\text{-}dt}(f),\ \log \mathrm{spar}(f)$$

All are polynomially related.

Thank You