When Hilbert approximates: A Strong Nullstellensatz for Approximate Polynomial Satisfiability

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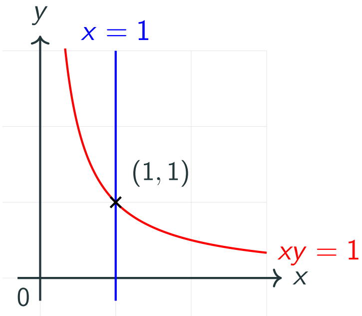
Exact Polynomial Satisfiability

Let $f = x - 1$ and $g = xy - 1$.

Common root at $(1, 1)$.



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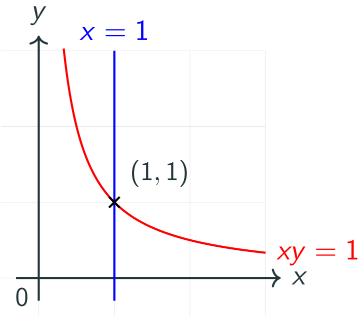
$0 \in \mathrm{Im}(\phi)$ where $\phi = (f, g)$.



Approximate Polynomial Satisfiability (APS)

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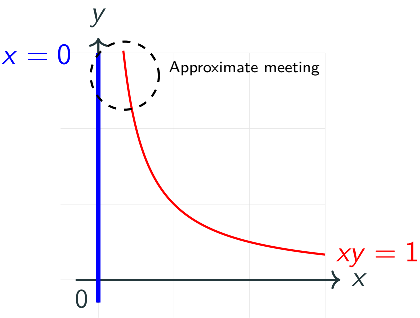
$$\implies f, g = 0 \mod \epsilon.$$



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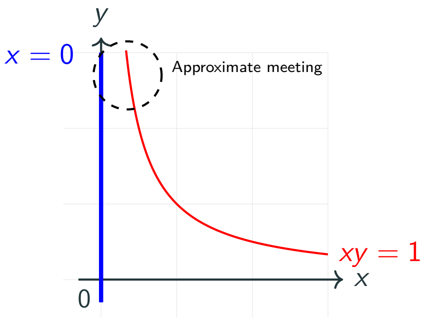

Vanishing Sets

For $\boldsymbol{f} = (f_1, \ldots, f_m) \in \mathbb{C}[x_1, \ldots, x_n]$, define

Exact vanishing set

$$V(\boldsymbol{f}) := \{a \in \mathbb{C}^n \mid f_i(a) = 0 \ \forall i\}.$$

Approximative vanishing set

$$\mathrm{AV}(\boldsymbol{f}) := \left\{\beta \in \mathbb{C}(\varepsilon)^n : f_i(\beta) = \varepsilon \frac{p_{i,\beta}(\varepsilon)}{q_{i,\beta}(\varepsilon)},\ q_{i,\beta}(0) \neq 0 \ \forall i\right\}.$$

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- APS asks whether $\mathrm{AV}(\boldsymbol{f})$ is non-empty.
- This captures "border" phenomena: limits of polynomial maps and border complexity.

Known starting point: Weak Approximative Nullstellensatz

Definition

Let $A(y_1, \ldots, y_m)$ be an annihilator of $\boldsymbol{f}$ if

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Classical WHN:

$$V(\boldsymbol{f}) = \emptyset$$

$$\iff 1 \in \langle f_1, \ldots, f_m \rangle.$$

Approximate analog (WAN):

$$\mathrm{AV}(\boldsymbol{f}) = \emptyset$$

$\iff \exists$ constant-1 annihilator.

Central Question

Classical Strong Hilbert's Nullstellensatz answers containment:

$$V(f_1, \dots, f_m) \subseteq V(g) \iff \exists r \in \mathbb{N},\ g^r = \sum_{i=1}^{m} f_i h_i$$

$$\text{for some } h_1, \dots, h_m \in \mathbb{C}[x_1, \dots, x_n]$$

$$\iff g \in \operatorname{rad}(\langle f_1, \dots, f_m \rangle).$$

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Approximate containment problem

Given $f_1,\dots,f_m,g\in\mathbb{C}[x_1,\dots,x_n]$, decide whether

$$\mathrm{AV}(f_1,\dots,f_m)\subseteq\mathrm{AV}(g).$$

Equivalently: every infinitesimal solution of f also makes g vanish infinitesimally.

Main theorem: Strong Approximative Nullstellensatz

Theorem (SAN)

$$\mathrm{AV}(g) \supseteq \mathrm{AV}(f_1, \ldots, f_m) \iff \exists r \in \mathbb{N},\ g^r = \sum_{i=0}^{r-1} \frac{F_i(f_1,\ldots,f_m)}{1+G_i(f_1,\ldots,f_m)}\, g^i$$

where $F_i, G_i \in \mathbb{C}[y_1, \ldots, y_m]$ *are constant-free.*

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Takeaway: From radicals, we shift to the new algebraic object of **integral closure** over a localized ideal.

Example

Let

$$f_1 = xy^2 - y + x^2 y, \qquad f_2 = xy - 1, \qquad g = x.$$

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SAN gives a *short certificate*:

$$g^2 = \frac{f_1}{1 + f_2}\, g - f_2.$$

Equivalence of WHN and SHN

Rabinowitsch's Trick

Over algebraically closed field k,

$$V(g) \supseteq V(f_1, \ldots, f_m) \iff V(f_1, \ldots, f_m, 1 - gy) = \emptyset.$$

Intuitively: g vanishes with f if and only if $1 - yg$ doesn't vanish simultaneously.

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- $V(f_1, \ldots, f_m, 1 - gy) = \emptyset$
  $\iff 1 = \sum_{i=1}^{m} f_i(\mathbf{x}) h_i(\mathbf{x}, y) + (1 - gy) h_{m+1}(\mathbf{x}, y).$

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WHN and SHN are also computationally equivalent.

Why classical Rabinowitsch does not transfer

g and $1 - yg$ can vanish simulataneously:

$$\text{If} \quad g(\beta) = \varepsilon \frac{p(\varepsilon)}{q(\varepsilon)}, \qquad p(\varepsilon) \neq 0$$

then choosing

$$y = \frac{q(\varepsilon)}{\varepsilon p(\varepsilon)}$$

makes $1 - yg$ vanish exactly.

Proof idea: approximate Rabinowitsch tricks

$\mathrm{AV}(\boldsymbol{f}) \subseteq \mathrm{AV}(g)$ is equivalent to two APS-infeasibility statements: whenever $\boldsymbol{f}$ goes to 0,

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No divergence

$f_1, \ldots, f_m, y, 1 - yg$
have no common APS.

No nonzero constant limit

$\forall c \in \mathbb{C}^\times : \quad f_1, \ldots, f_m, g - c$
have no common APS.

Flow of the proof

$f_1, \ldots, f_m, y, 1 - yg$
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$\forall c \in \mathbb{C}^\times,$
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Goal: $g^r = \sum_{i=0}^{r} g^i F_i(\boldsymbol{f})$
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$A'(\boldsymbol{f}, g) = 0$
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algebraic manipulation

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Complexity of SAN

Theorem (GSS18)

WAN is in PSPACE.

Decision Problem of SAN

When the input polynomials are given by algebraic circuits, deciding whether $\mathrm{AV}(f_1, \ldots, f_m) \subseteq \mathrm{AV}(g)$.

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Recall that WHN and SHN were computationally equivalent.

But the reduction from SAN to WAN involves roots of a univariate polynomial, which may not be computable.

Algorithmic result: SAN is in PSPACE

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We essentially prove "effective" versions of our approximate Rabinowitsch tricks using:

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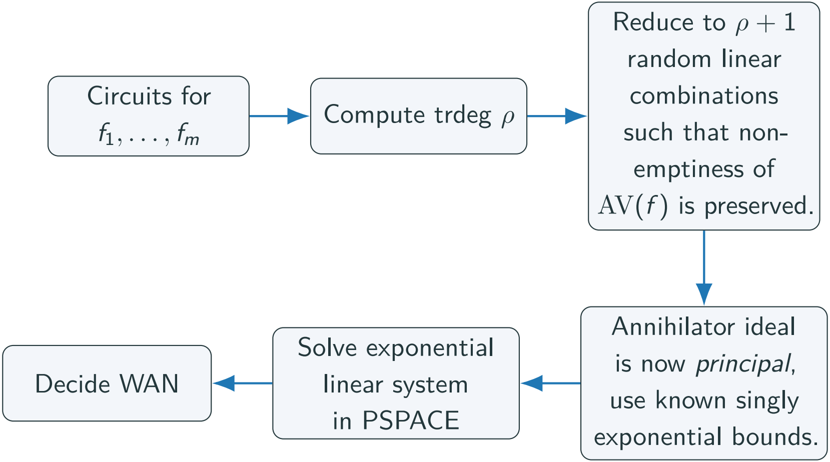
to obtain an exponential-sized linear algebra system, which can be solved in PSPACE.

Use of GSS random reductions



We use this framework on our system of WAN-infeasibilities to get a singly exponential bound on the degree of the SAN certificate.

Conclusion and open directions

1. WAN is the weak Nullstellensatz for APS; SAN is the strong version.
2. Approximate containment is governed by integral closure over a localized ideal.
3. Despite apparent EXPSPACE barriers, SAN has a PSPACE decision algorithm.

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- Define a useful notion of dimension for approximative vanishing sets.
- Can we improve complexity of WAN? Put it in CH or even PH?
- What about generators of the integral closure in SAN? Can we algorithmically obtain them?

Thank You!

Questions?