Frontier Space-Time Algorithms Using Only Full Memory

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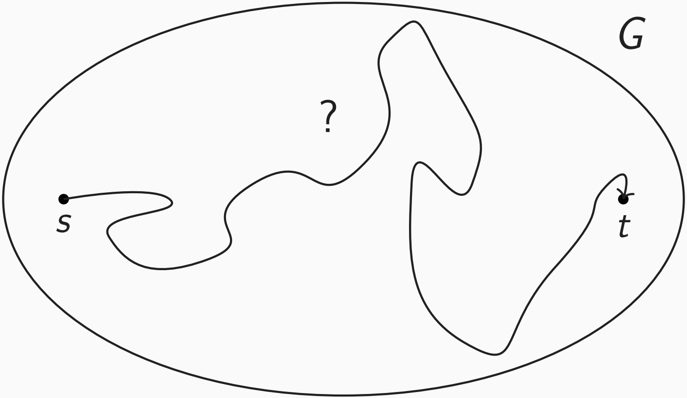
The problem

STCONN: given a directed graph G and $s, t \in V(G)$, decide whether there is a path from s to t.



n = number of vertices, m = number of edges

What do we know about STCONN

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- Edmonds, Poon, Achlioptas'99: matching space lower bound for poly(n)-time for pebbling algorithms (NNJAG)

Catalytic space

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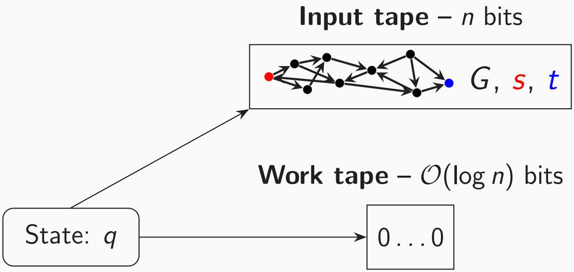
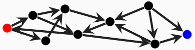
- Cook, Mertz'24: Tree Evaluation in $\mathcal{O}(\log n \cdot \log\log n)$ space
- Williams'25 using Cook, Mertz'24: For multitape Turing machines, $\mathrm{TIME}[t] \subseteq \mathrm{SPACE}[\sqrt{t \log t}]$

Catalytic Turing machine [BCKLS'14]



Catalytic Turing machine [BCKLS'14]

Input tape – n bits



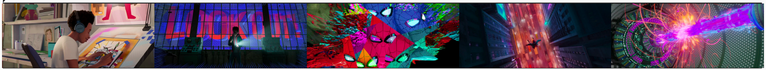
G, s, t

Work tape – $\mathcal{O}(\log n)$ bits

State: q

$0 \ldots 0$

Catalytic tape – poly(n) bits



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Our results

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	Free space	Catalytic space	Time
BCKLS'14	$\mathcal{O}(\log n)$	$\text{poly}(n)$	$\text{poly}(n)$
Cook, Pyne'26	$\mathcal{O}(\log n)$	$\tilde{\mathcal{O}}(n^2)$	$\tilde{\mathcal{O}}(n^3 m)$
Theorem 1	$\mathcal{O}(\log n)$	$n/2^{\Omega(\sqrt{\log n})}$	$\text{poly}(n)$

$n =$ number of vertices, $m =$ number of edges

Our results, randomized

Theorem $1+\varepsilon$
STCONN can be solved with a randomized algorithm in $\mathcal{O}(\log n)$ free space, $n/2^{\Omega(\sqrt{\log n})}$ catalytic space and $\mathcal{O}(n^{2+\varepsilon}m)$ time.

	Free space	Catalytic space	Time
Cook, Pyne'26	$\mathcal{O}(\log n)$	$\tilde{\mathcal{O}}(n)$	$\tilde{\mathcal{O}}(nm)$
Theorem $1+\varepsilon$	$\mathcal{O}(\log n)$	$n/2^{\Omega(\sqrt{\log n})}$	$\mathcal{O}(n^{2+\varepsilon}m)$

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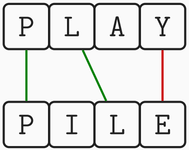
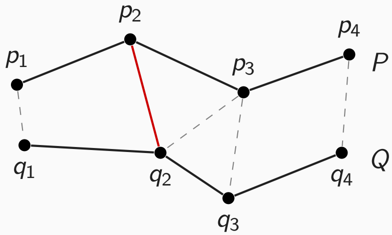
Other problems

- Three other problems:
  - Edit distance (ED)
  - Longest common subsequence (LCS)
  - Discrete Fréchet distance (DFD)





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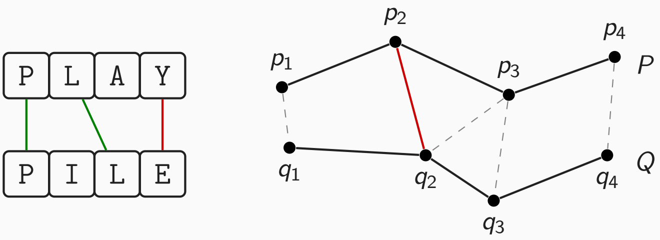

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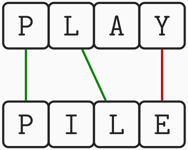

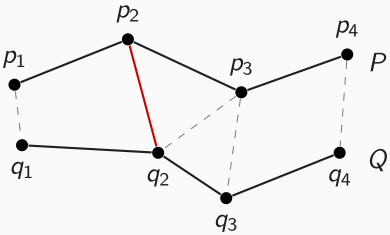

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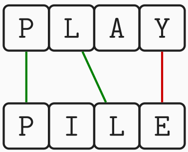

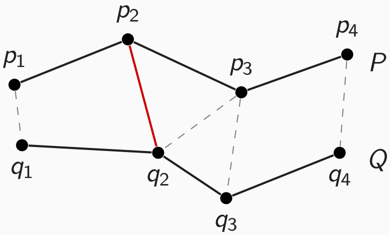

- All reduce to finding (weighted) longest/shortest/any path in a "grid" graph – $\Theta(n^2)$ *blowup*
- Kiyomi, Horiyama, Otachi'21: $n/2^{\Omega(\sqrt{\log n})}$-space poly-time algorithms for ED and LCS.

Solving ED, LCS, DFD

Theorem 2
ED, LCS and DFD can be solved in $\mathcal{O}(\log n)$ free space, $n/2^{\Omega(\sqrt{\log n})}$ catalytic space and $\mathrm{poly}(n)$ time.

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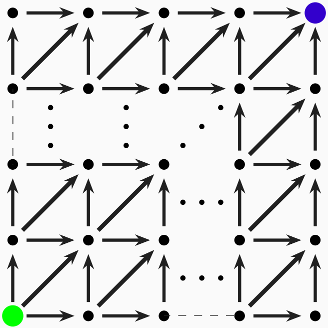

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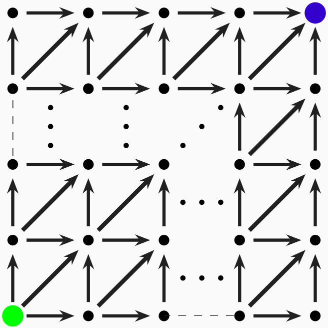

- The graph has $\Theta(n^2)$ vertices
- Need to compute the *shortest* path

BBRS: Long paths

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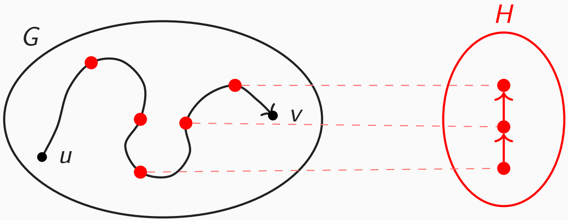

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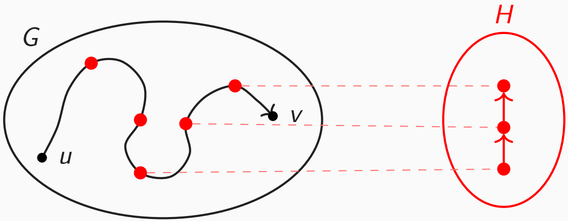

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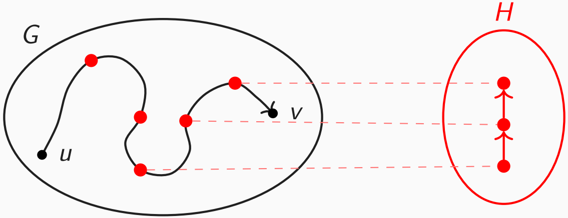

Choose $\lambda := 2^{\Theta(\sqrt{\log n})}$, find a set S of n/λ vertices such that

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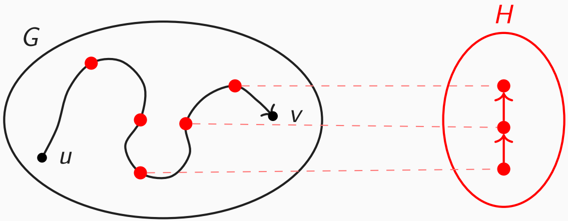

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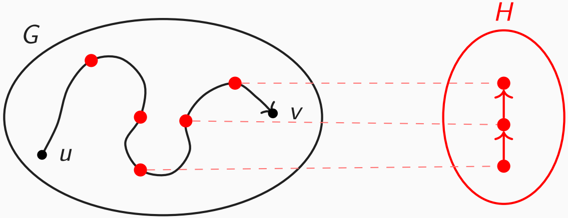

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The only remaining query:

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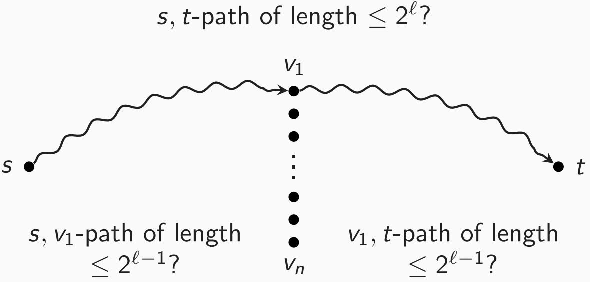

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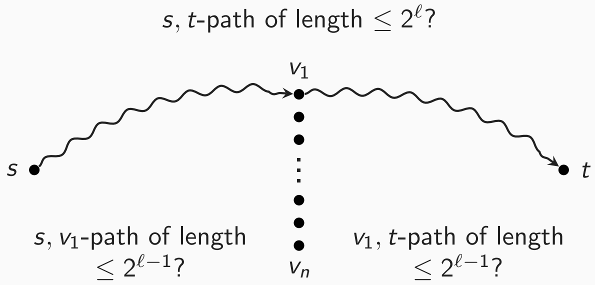

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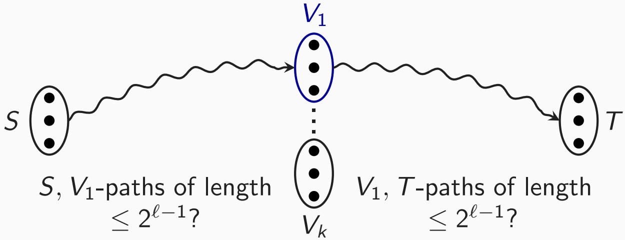

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Batch size $n/\lambda = n/2^{\Theta(\sqrt{\log n})}$

Space: $\mathcal{O}(\log n \cdot \log(\text{distance}))$
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Batched: $n/2^{\Omega(\sqrt{\log n})}$
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- Short-path algorithm: "Is there a λ-length path from u to v in G?"

Applying BBRS with catalytic space

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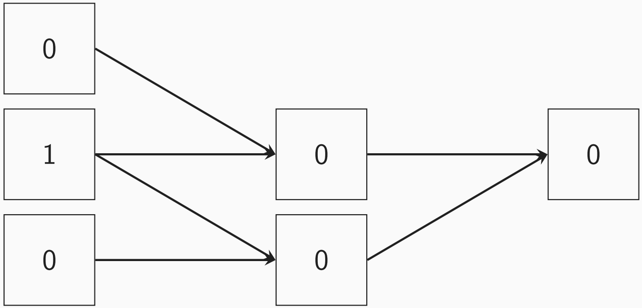

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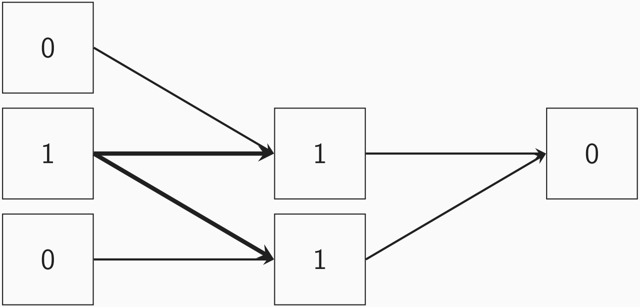

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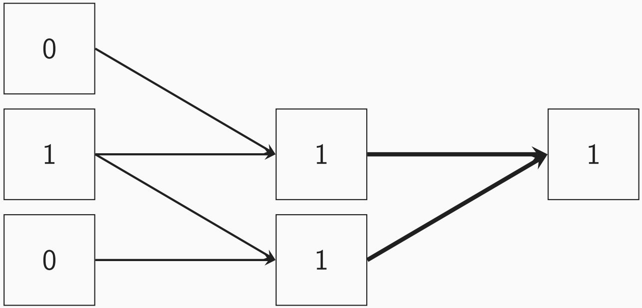
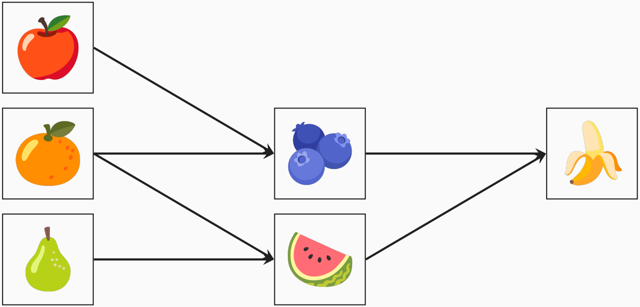


Oh no



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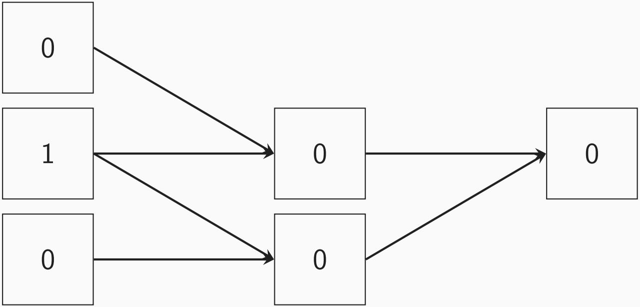

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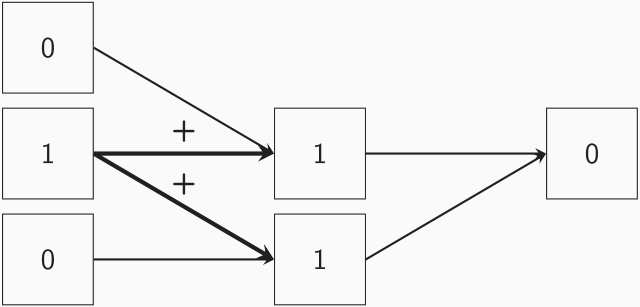

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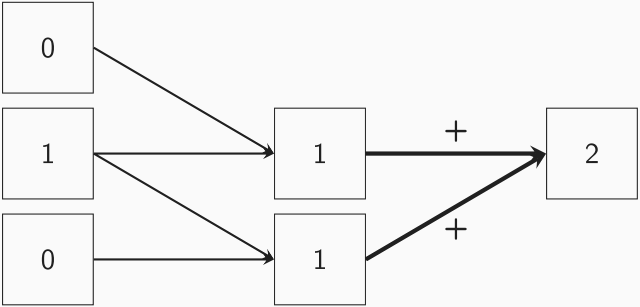

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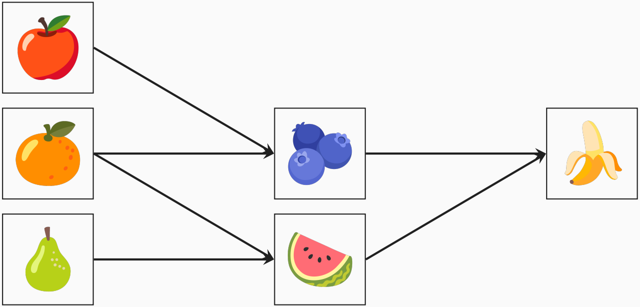

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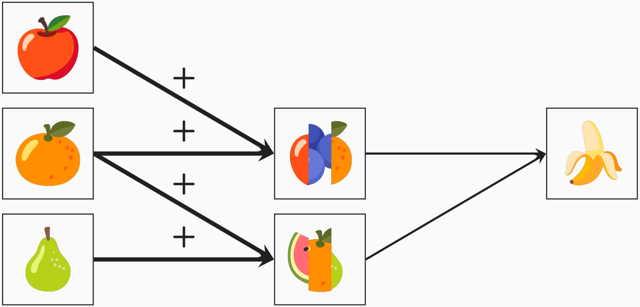

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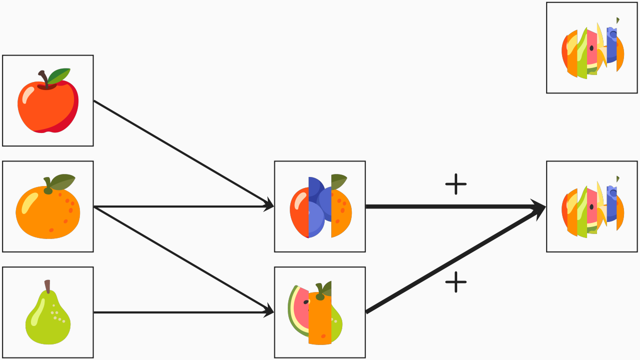

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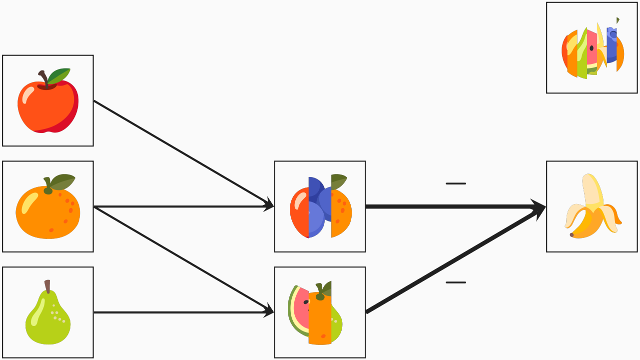

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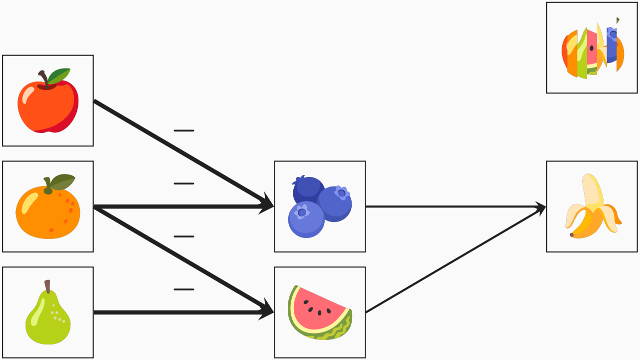

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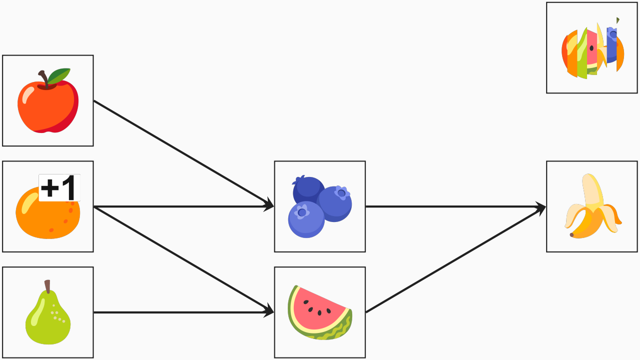

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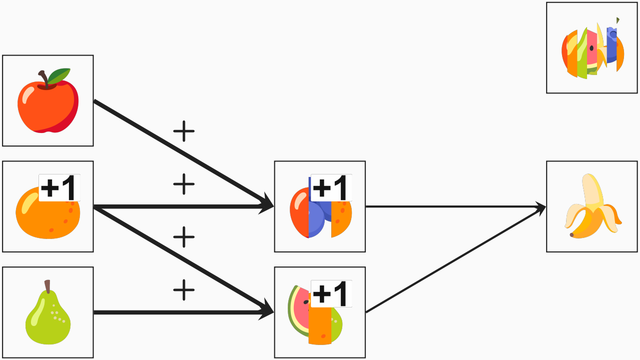

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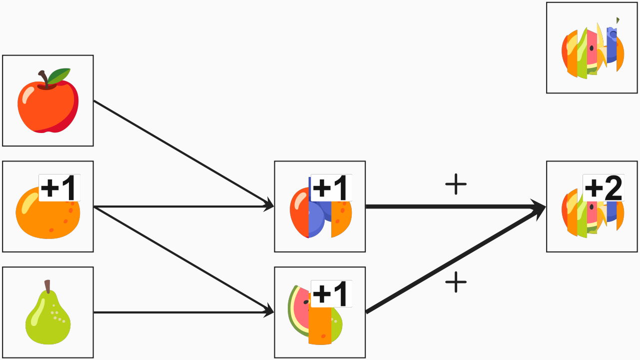

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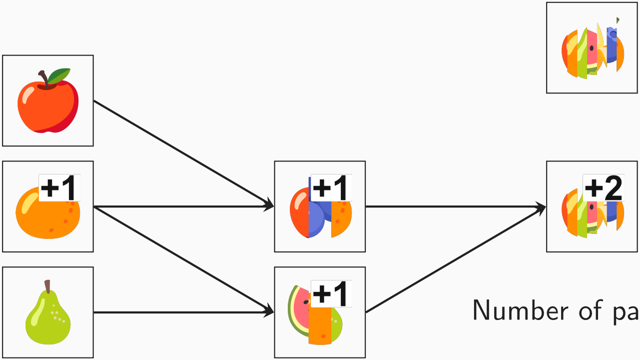

Number of paths: 2

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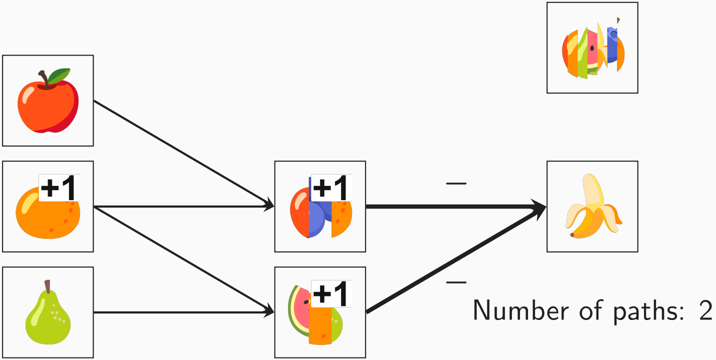

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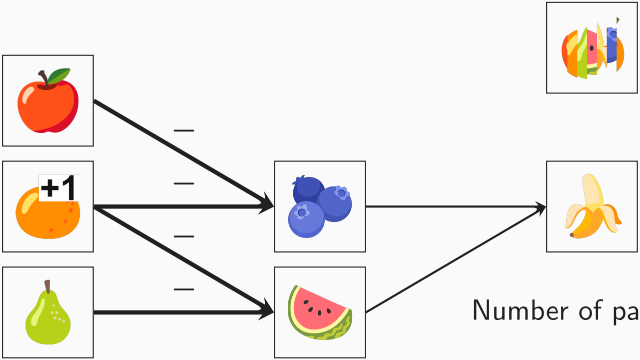

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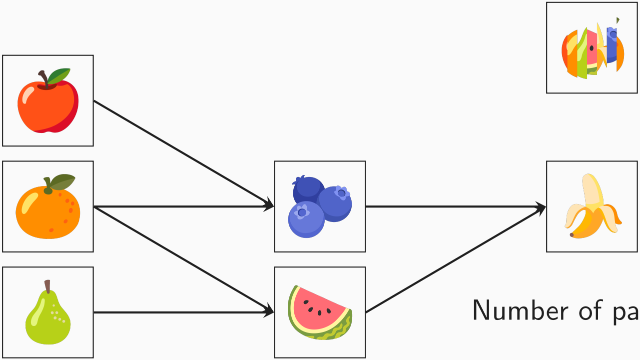

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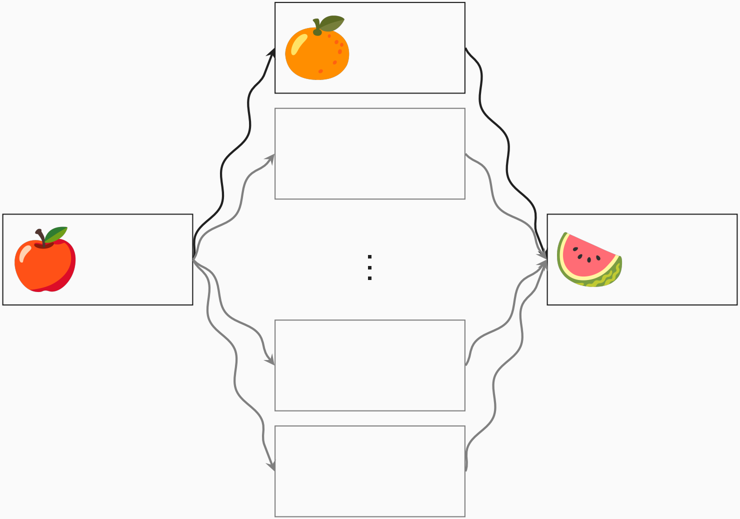
Applying BBRS with catalytic space

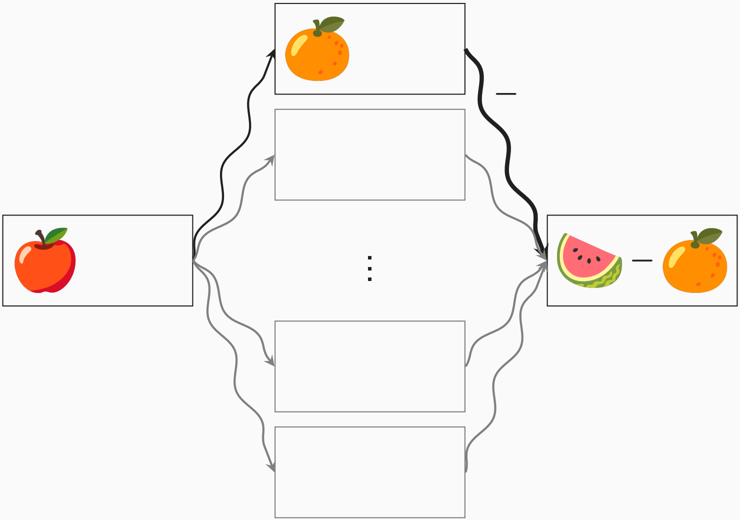
Reminder of Theorem 1
STCONN can be solved in $\mathcal{O}(\log n)$ free space, $n/2^{\Omega(\sqrt{\log n})}$ catalytic space and poly(n) time.

1. **Build a "simply constructible" representative set** U ✓
2. Long walks: catalytic BFS – Cook, Pyne'26 ✓
3. **Short walks: catalytic Savitch** – concurrently Edenhofer'26

Catalytic Savitch



Catalytic Savitch



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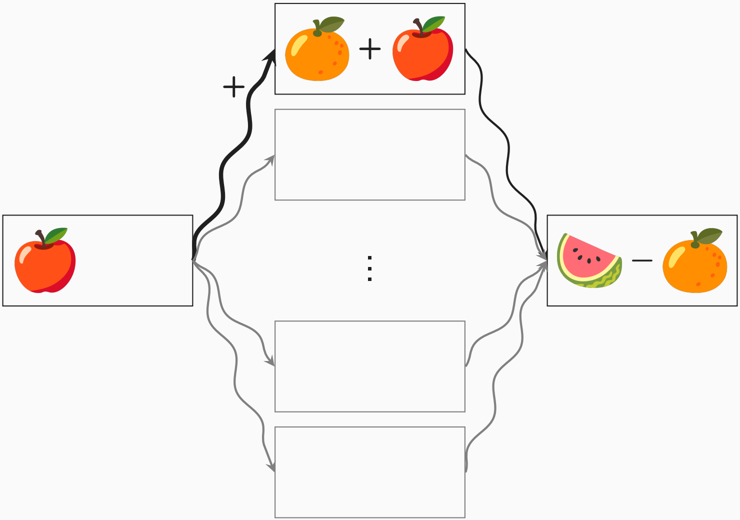

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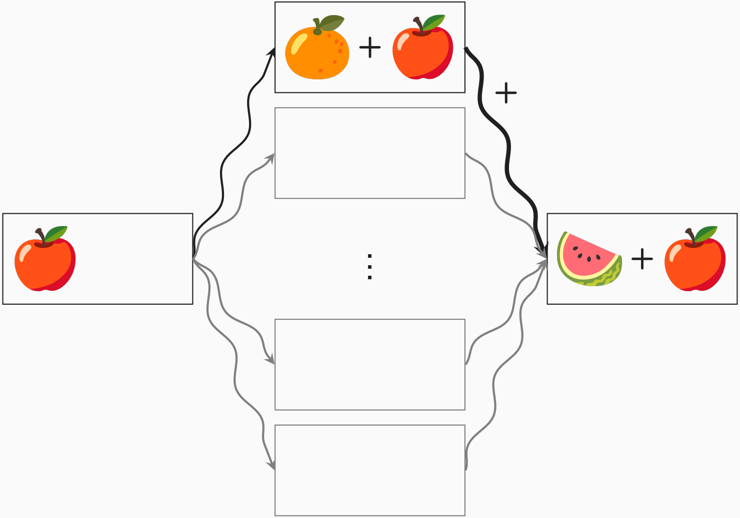

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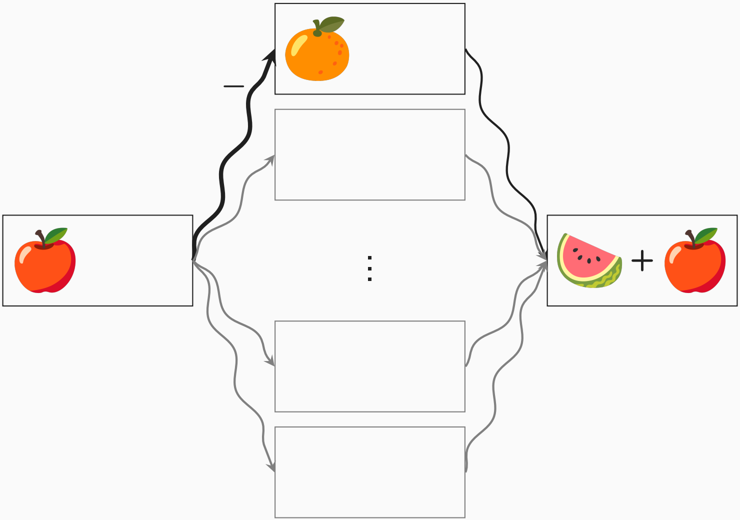

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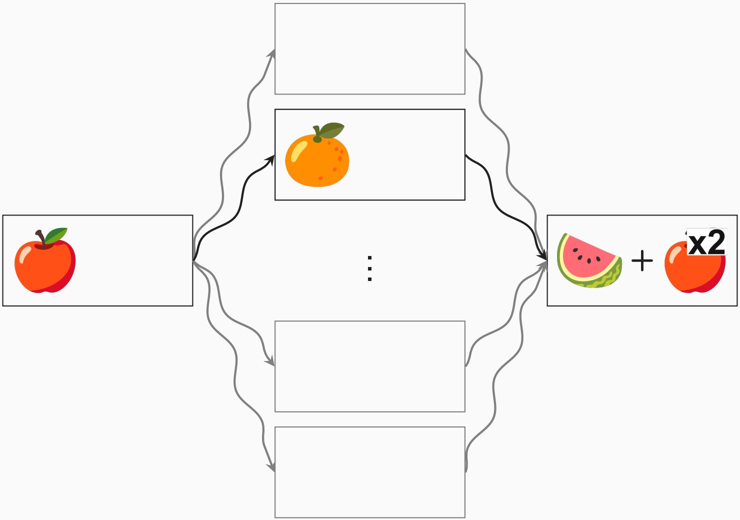

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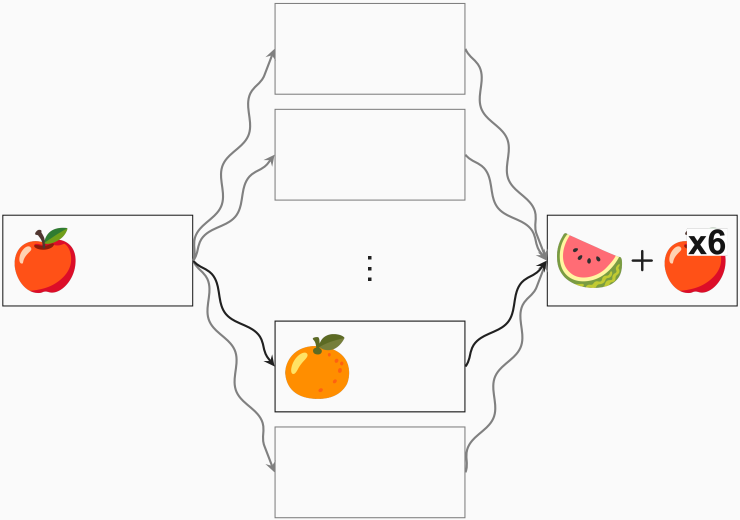

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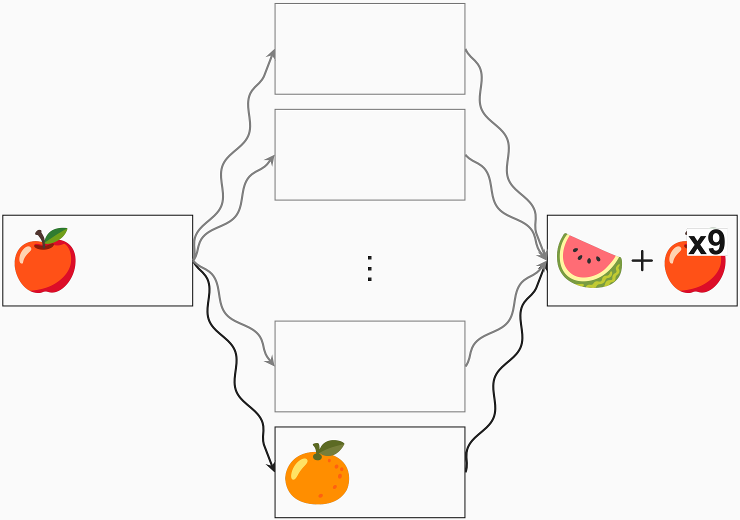

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Open questions

- A poly-time algorithm using space smaller than $n/2^{\Theta(\sqrt{\log n})}$?

Open questions

- A poly-time algorithm using space smaller than $n/2^{\Theta(\sqrt{\log n})}$?
- An interesting double-sided error algorithm for STCONN?

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Thank you!