RESOLUTION WIDTH LIFTS TO NEAR-QUADRATIC-DEPTH RES($\oplus$) SIZE

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PLAN:

- RESOLUTION
- RESOLUTION OVER PARITIES
- KNOWN RESULTS
- NEW RESULTS
- PROOF IDEAS (?)

BASICS

- WE CONSIDER UNSATISFIABLE CNFs φ:

  $(x \vee y) \wedge (\neg x \vee \neg y) \wedge (y \vee z) \wedge (\neg y \vee \neg z) \wedge (x \vee z) \wedge (\neg x \vee \neg z)$

- WE STUDY SIZE OF REFUTATIONS

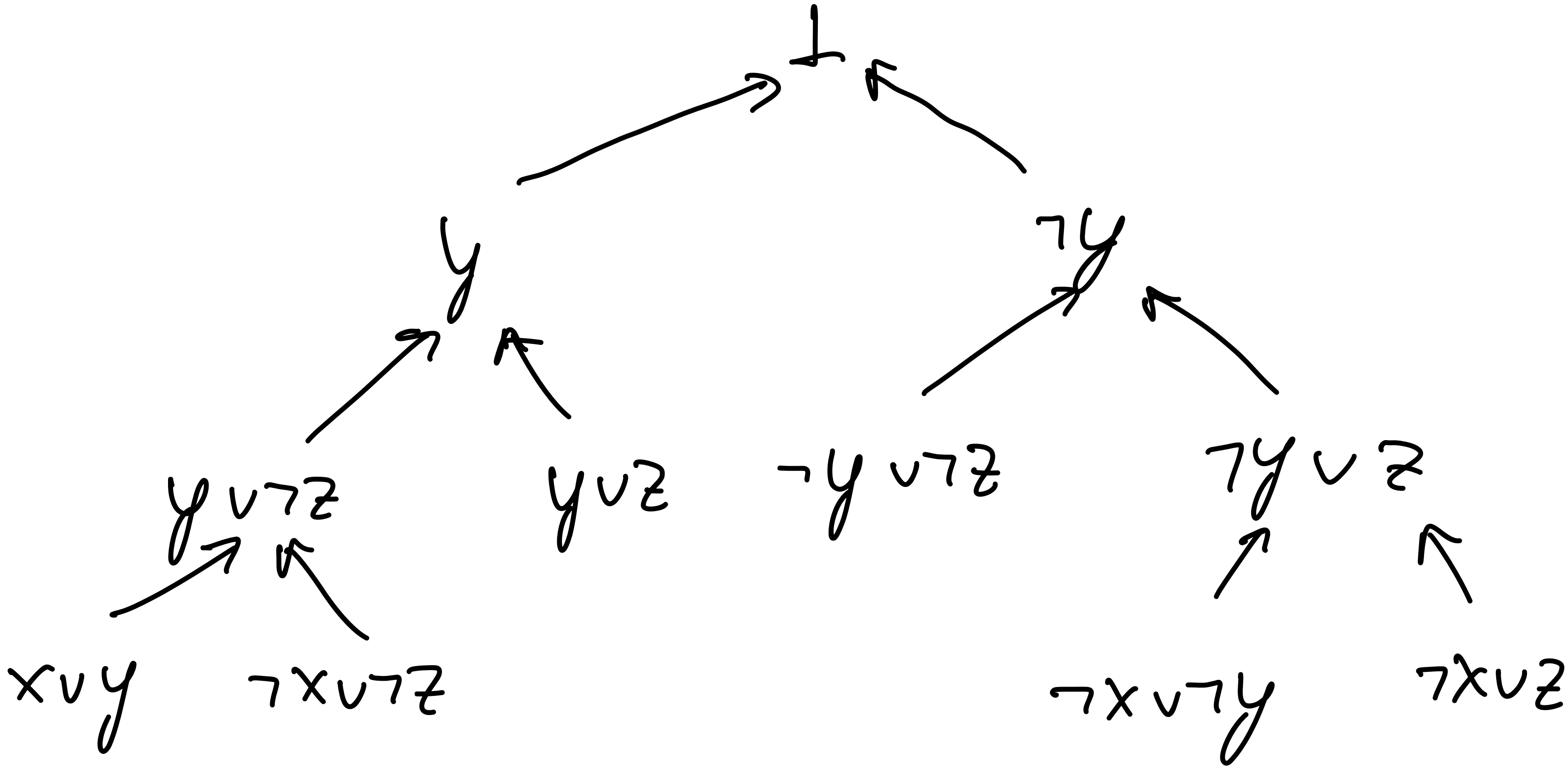
- INPUT SIZE PARAMETERS
  - NUMBER OF INPUT VARIABLES n
  - FORMULA SIZE $|\varphi|$
- NORMALLY POLYNOMIALLY RELATED

RESOLUTION

- START WITH CLAUSES OF φ: $C_1, \dots, C_k$
- REPEATEDLY APPLY THE FOLLOWING RULES
  - RESOLUTION RULE: $\dfrac{A \vee x \quad , B \vee \neg x}{A \vee B}$
  - WEAKENING RULE: $\dfrac{C}{C \vee D}$
- DEDUCE EMPTY CLAUSE $\perp$
- PROOF CAN BE VIEWED AS A DAG
  - SOURCES AT CLAUSES OF φ
  - SINK AT $\perp$

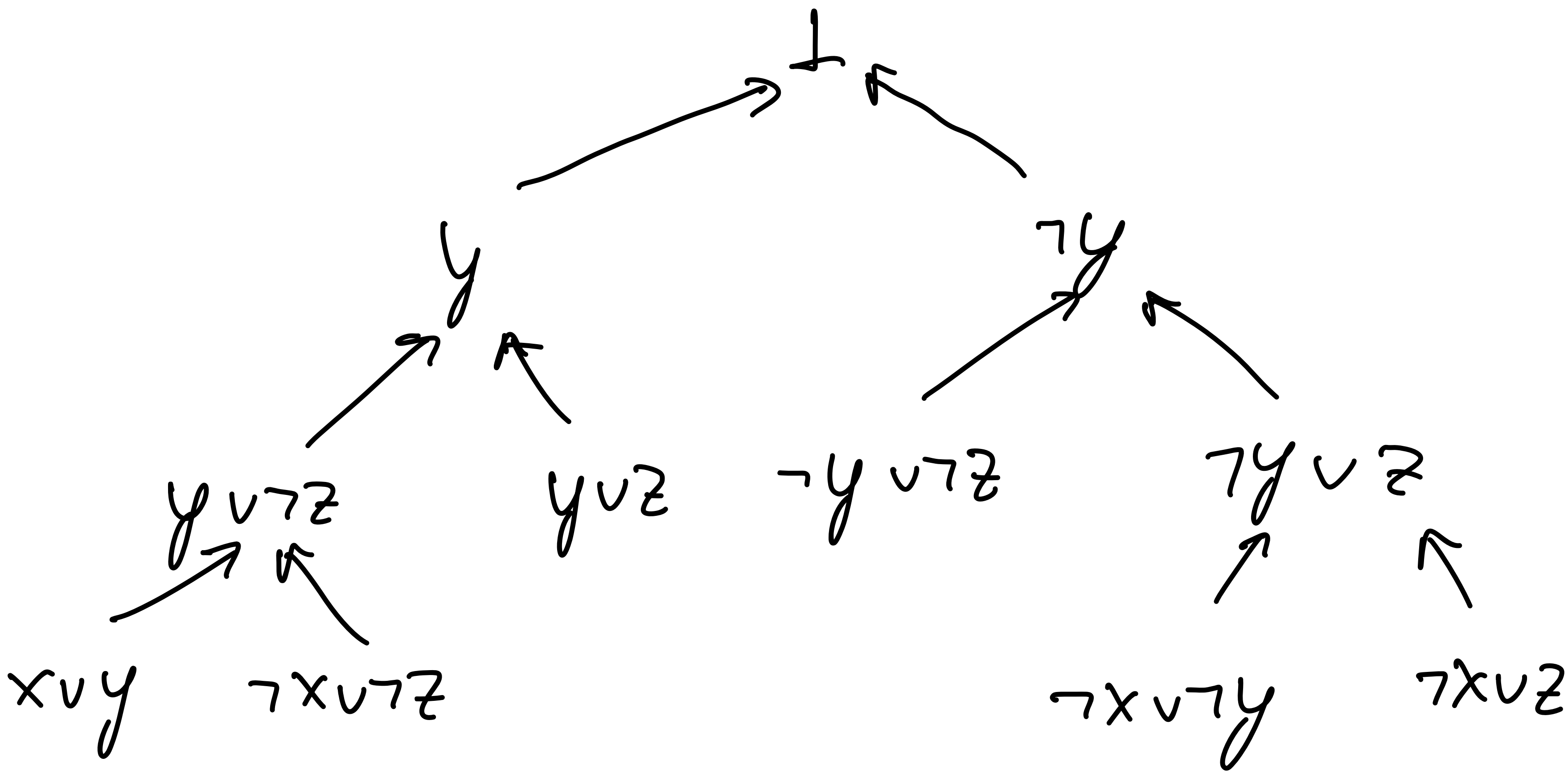
RESOLUTION: EXAMPLE

$$(x \vee y) \wedge (\neg x \vee \neg y) \wedge (y \vee z) \wedge (\neg y \vee \neg z) \wedge (x \vee z) \wedge (\neg x \vee \neg z)$$



RESOLUTION: EXAMPLE

$$(x \lor y) \land (\neg x \lor \neg y) \land (y \lor z) \land (\neg y \lor \neg z) \land (x \lor z) \land (\neg x \lor \neg z)$$



COMPLEXITY MEASURES:

- MINIMAL SIZE OF PROOF $S_{RES}(\varphi)$
- MINIMAL WIDTH OF PROOF $S_w(\varphi)$
- MINIMAL DEPTH OF PROOF $S_d(\varphi)$

RESTRICTED RESOLUTION

- TREE-LIKE VERSION: DAG IS A TREE

- REGULAR RESOLUTION:
  IF IN VERTEX v WE RESOLVE x,
  THEN x WAS NOT RESOLVED IN THE
  SUBGRAPH WITH SINK IN x

RESTRICTED RESOLUTION

- TREE-LIKE VERSION: DAG IS A TREE

- REGULAR RESOLUTION:
  IF IN VERTEX v WE RESOLVE x,
  THEN x WAS NOT RESOLVED PRIOR TO v

$$
\begin{array}{c}
\text{RES} \\
| \\
\text{REG RES} \\
| \\
\text{TREE RES}
\end{array}
$$

RESOLUTION LOWER BOUNDS

- TSEITIN '68: EXPONENTIAL LOWER BOUND FOR REG RES
- HAKEN '85: EXPONENTIAL LOWER BOUND FOR RES
- BEN-SASSON, WIGDERSON '99:
  IF φ IS UNSAT CNF OF WIDTH k, THEN:
  - $S_{TREE\text{-}RES}(\varphi) \geq 2^{w_{RES}(\varphi) - k}$
  - $S_{RES}(\varphi) \geq 2^{\Omega\left(\frac{(w_{RES}(\varphi) - k)^2}{n}\right)}$

RESOLUTION OVER PARITIES RES($\oplus$)

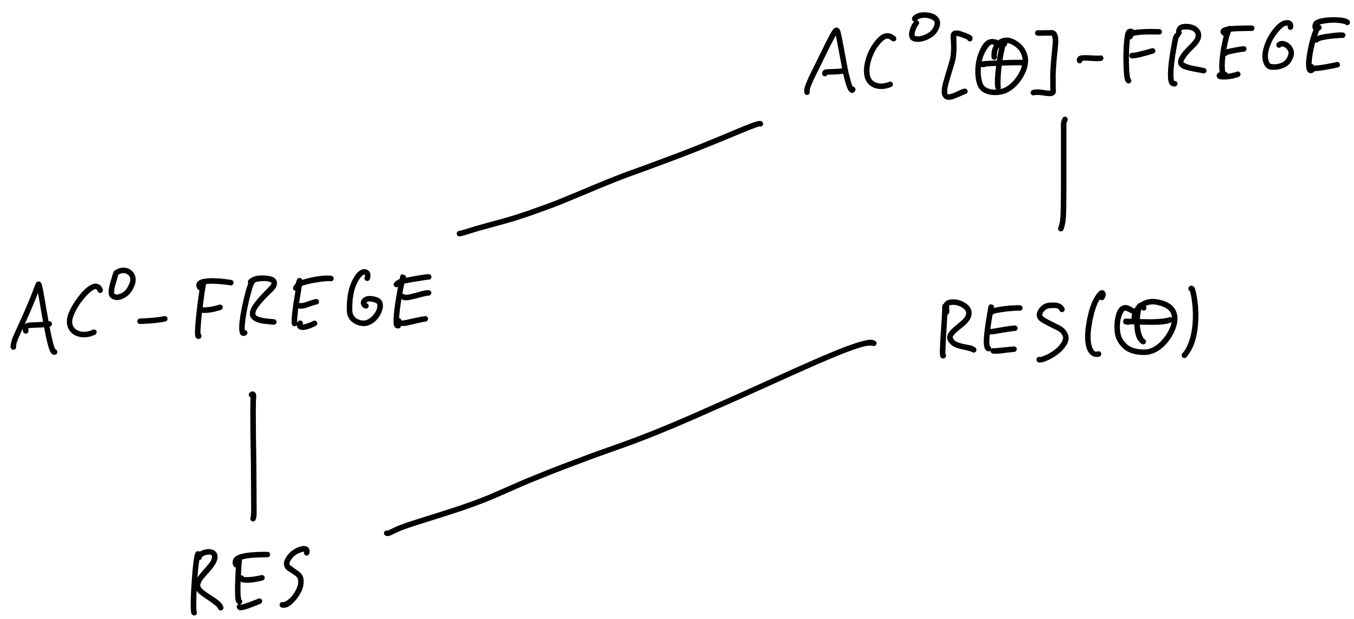
- START WITH CLAUSES OF φ: $C_1, \dots, C_k$
- DEDUCE EMPTY CLAUSE $\bot$
- LINES OF PROOF: ORs OF $\mathbb{F}_2$ LINEAR EQUATIONS

$$(x+y=0) \vee (y+z=1)$$

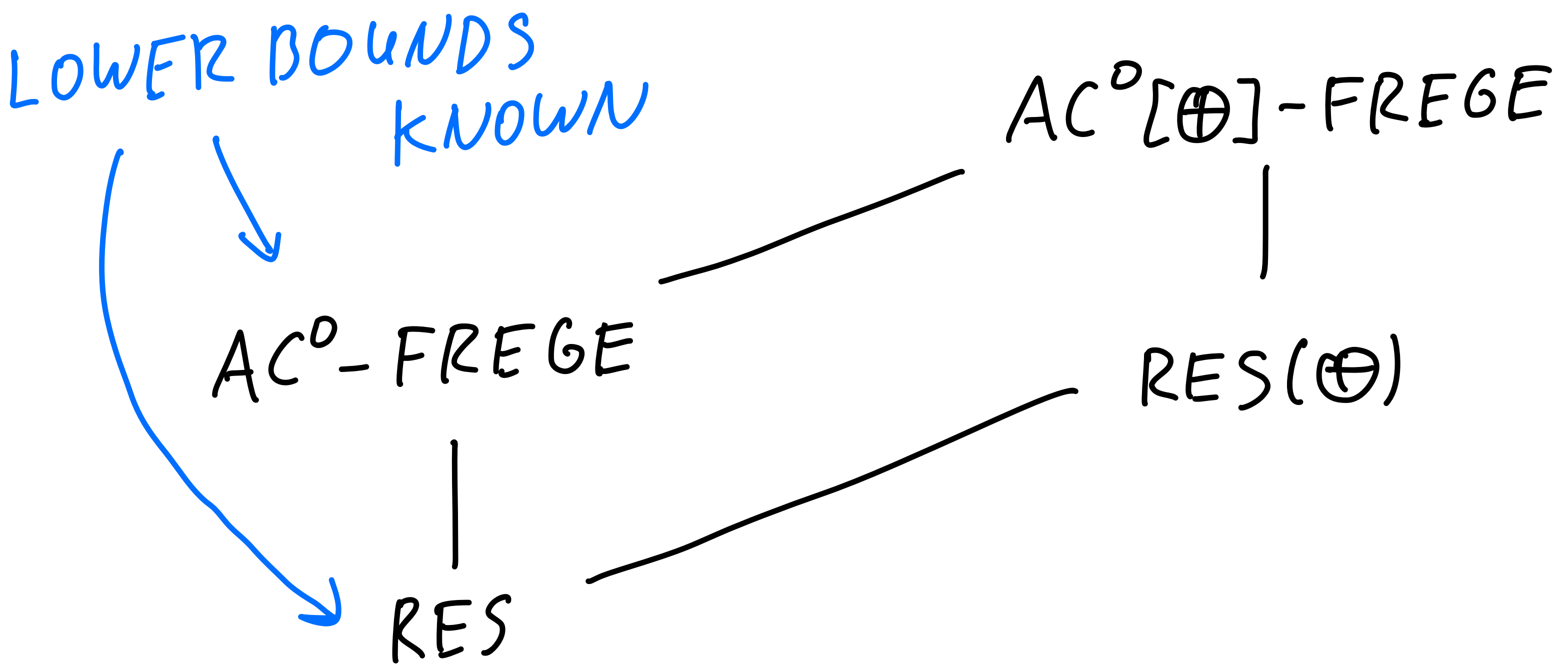
- RULES:
  - RESOLUTION RULE: $\dfrac{A \vee (f=0), \; B \vee (f=1)}{A \vee B}$
  - WEAKENING RULE: $\dfrac{C}{D}$, IF C SEMANTICALLY IMPLIES D

- OPEN PROBLEM: SUPERPOLYNOMIAL LOWER BOUND FOR RES($\oplus$)

BIGGER PICTURE



BIGGER PICTURE



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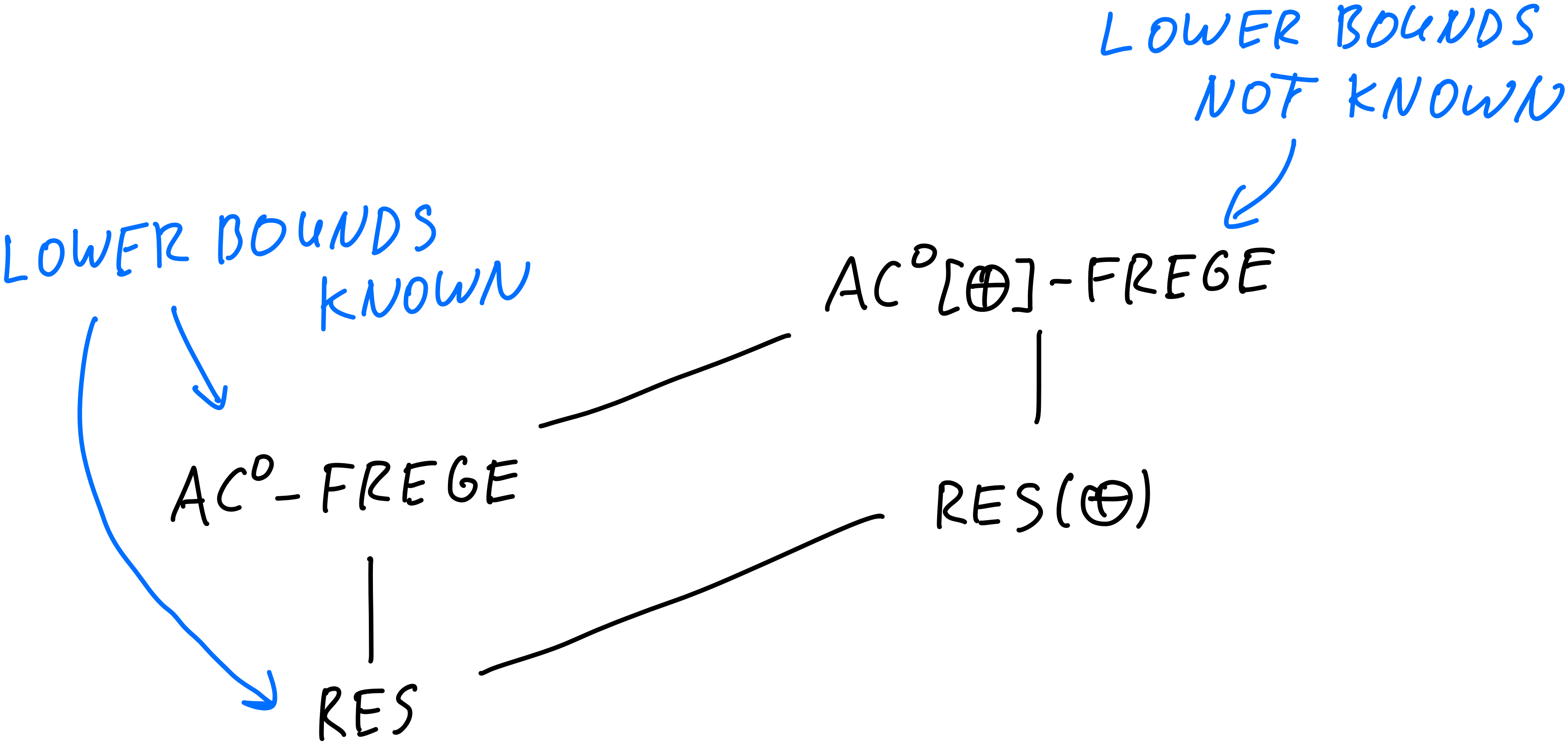

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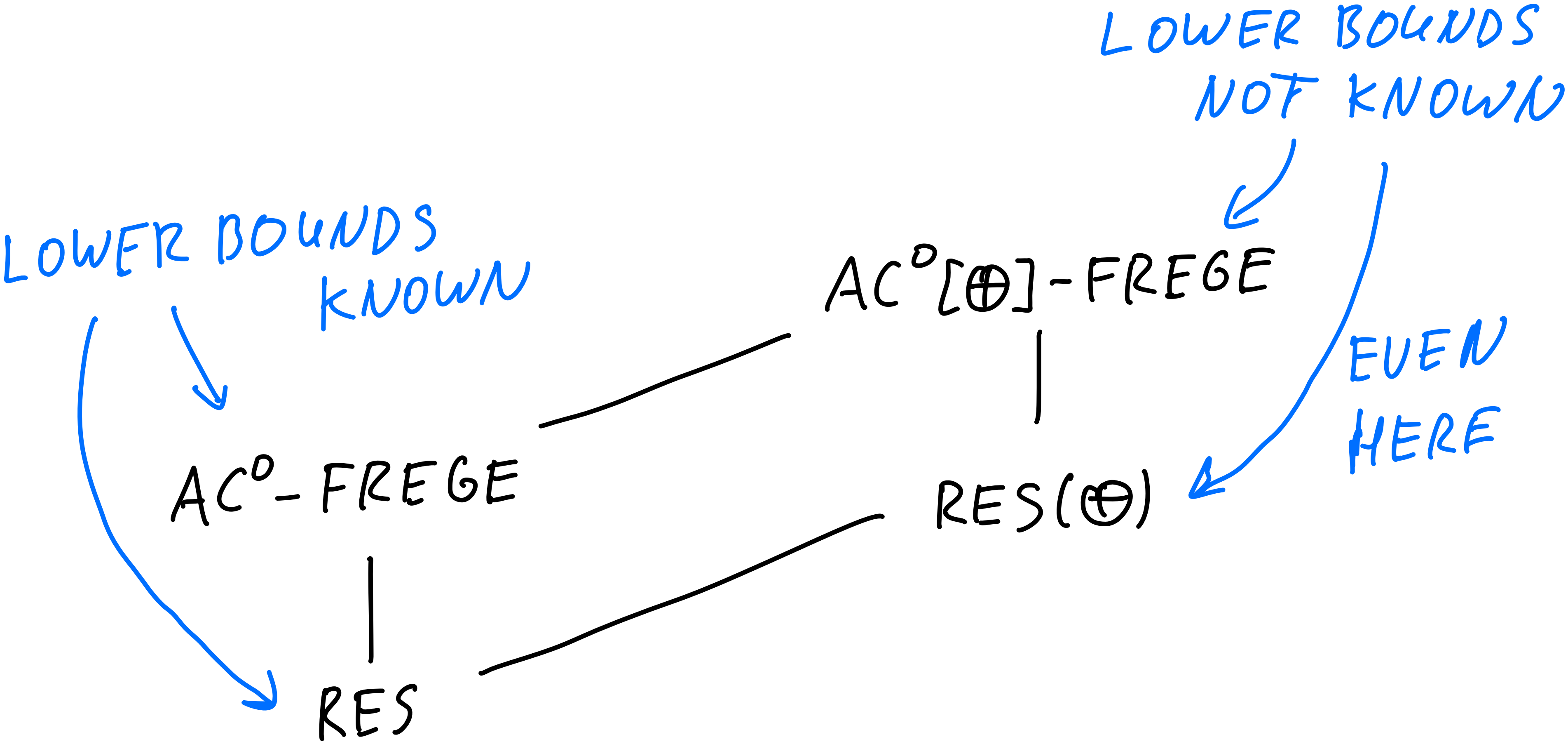

RESTRICTED RES($\oplus$)

- TREE-LIKE RES($\oplus$): DAG IS A TREE

- REGULAR RES($\oplus$):
  IF IN VERTEX v WE RESOLVE f,
  THEN f IS NOT IN SPAN OF LINEAR FORMS
  RESOLVED PRIOR TO v

- DEPTH-d RES($\oplus$):
  DAG IS OF DEPTH
  AT MOST d

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RES($\oplus$)
|
DEPTH-d RES($\oplus$), $d \geq n$
|
REG RES($\oplus$)
|
TREE RES($\oplus$)

KNOWN LOWER BOUNDS

- (ITSYKSON, SOKOLOV, '14):
  FIRST LOWER BOUND FOR TREE RES($\oplus$)

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  (BEAME, KOROTH, '23)
  LOWER BOUND FOR TREE RES($\oplus$) VIA LIFTING

LIFTING

- $\varphi(y_1, \ldots, y_m)$ - UNSAT CNF
- $g : \{0,1\}^\ell \to \{0,1\}$ - GADGET
- LIFTED FORMULA:

$$\varphi \circ g = \varphi(g(x_{11}, \ldots, x_{1\ell}), \ldots, g(x_{m1}, \ldots, x_{m\ell}))$$

CONVERTED INTO CNF

- LIFTING THEOREM:
  - ASSUME φ IS HARD IN WEAK PROOF SYSTEM
  - ASSUME SOME PROPERTIES OF g
  - THEN $\varphi \circ g$ IS HARD FOR STRONGER PROOF SYSTEM

CMSS '23 LIFTING THEOREM

- g IS <u>K-STIFLING</u> IF
  $\forall b \in \{0,1\}, \forall S \subseteq [\ell], |S| = \ell - k$
  WE CAN FIX VARIABLES IN S IN SUCH A WAY
  THAT $g|_{[\ell]\setminus S} \equiv \alpha$
- EXAMPLE: MAJ_{2k+1} IS K-STIFLING
- [CMSS '23]:

  g IS K-STIFLING $\Rightarrow$ $S_{TREE\text{-}RES}(\varphi \circ g) \geq 2^{d_{RES}(\varphi) \cdot k}$

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  LOWER BOUND FOR TREE RES($\oplus$) VIA LIFTING

- (EFREMENKO, GARLIC, ITSYKSON, '24)
  LOWER BOUND FOR REG RES($\oplus$)

LOWER BOUNDS FOR BOUNDED DEPTH

- (ALEKSEEV, ITSYKSON, '25)
  EXPONENTIAL LOWER BOUND FOR
  $n \log n \log n$ - DEPTH RES($\oplus$)

LOWER BOUNDS FOR BOUNDED DEPTH

- (ALEKSEEV, ITSYKSON, '25) + (EFREMENKO, ITSYKSON '25)
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  <u>HOW:</u> LIFTING OF TSEITIN TAUTOLOGIES VIA 2-STIFLING GADGET

LOWER BOUNDS FOR BOUNDED DEPTH

- (ALEKSEEV, ITSYKSON, '25) + (EFREMENKO, ITSYKSON '25)
  EXPONENTIAL LOWER BOUND FOR
  $n \log n$ - DEPTH RES($\oplus$)
  HOW: LIFTING OF TSEITIN TAUTOLOGIES VIA 2-STIFLING GADGET

- (ITSYKSON, KNOP, '26)
  EXPONENTIAL LOWER BOUND FOR
  $n \log n$ - DEPTH RES($\oplus$)
  HOW: LIFTING FROM $W_{RES}(\varphi)$ WITH $\log n$-SIZE GADGET

LOWER BOUNDS FOR BOUNDED DEPTH

- (BHATTACHARYA, BYRAMJI, CHATTOPADHYAY, IMPAGLIAZZO '26)

  LOWER BOUND OF $2^{n^{\varepsilon}}$ FOR $n^{2-\varepsilon}$-DEPTH RES($\oplus$)

  <u>HOW</u>: LIFTING FROM $(\Omega(n), \Omega(n))$-DT HARD CNF VIA 1-STIFLING GADGET

- (ALEXEEV, GAEVOY '26)

  ESSENTIALLY SAME BOUND,
  NEW FAMILY OF FORMULAS

NEW BOUND: LIFTING FROM $w_{RES}(\varphi)$

- WE PROVE: ESSENTIALLY THE SAME BOUND BUT LIFTING FROM $w_{RES}(\varphi)$ VIA 1-STIFLING GADGET

MORE PRECISELY,

<u>THEOREM.</u> IF g IS 1-STIFLING, THEN $\forall \varphi$ ANY RES($\oplus$) REFUTATION OF $\varphi \circ g$ OF SIZE S HAS DEPTH $\Omega(w_{RES}(\varphi)^2 / \log S)$

CONSEQENCE: SUPERCRITICAL TRADEOFF

(ITSYKSON, KNOP, '25):

$\exists\, \varphi_n$ OF SIZE $n^{O(\log^2 n)}$ SUCH THAT

- $S_{RES}(\varphi_n) = n^{O(\log^2 n)}$, $d_{RES}(\varphi_n) \leq n$
- ANY RES($\oplus$) REFUTATION OF φ_n OF SIZE $2^{O(\frac{n}{\log^4 n})}$ HAS DEPTH $\Omega(n \log n)$

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<u>THEOREM</u>. $\exists\, \varphi_n$ OF SIZE $poly(n)$ SUCH THAT

- $S_{RES}(\varphi_n) = poly(n)$, $d_{RES}(\varphi_n) \leq n$
- ANY RES($\oplus$) REFUTATION OF φ_n OF SIZE S HAS DEPTH $\Omega(n^2/\log^2 n \log S)$

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<u>THEOREM</u>. $\exists \varphi_n$ OF SIZE <u>poly(n)</u> SUCH THAT

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THEOREM. $\exists\, \varphi_n$ OF SIZE poly(n) SUCH THAT

- $S_{RES}(\varphi_n) = \text{poly}(n)$, $d_{RES}(\varphi_n) \le n$
- ANY RES($\oplus$) REFUTATION OF φ_n OF SIZE S HAS DEPTH $\Omega(n^2 / \log^2 n \log S)$

LOWER BOUNDS FOR UNRESTRICTED RES($\oplus$)

- (KHANIKI '22) $\Omega\left(\frac{n^2}{\log^{\log\log n} n}\right)$ LOWER BOUND ON SIZE OF RES($\oplus$) WITH SYNTACTIC WEAKENING RULE

- (BBCI '26) IMPLIES $\Omega\left(\frac{n^2}{\log n}\right)$ LOWER BOUND ON SIZE OF RES($\oplus$)

<u>THEOREM</u>. φ – UNSAT CNF, g – 1-STIFLING

$\Rightarrow S_{RES(\oplus)}(\varphi \circ g) = \Omega(w_{RES}(\varphi)^2)$

<u>COROLLARY</u>. $\exists$ $O(1)$-CNF φ OF SIZE $O(n)$ WITH $S_{RES}(\varphi) = \Omega(n^2)$

LOWER BOUNDS FOR UNRESTRICTED $\mathrm{RES}(\oplus)$

THEOREM φ - UNSAT CNF, g - 1-STIFLING

$\Rightarrow S_{\mathrm{RES}(\oplus)}(\varphi \circ g) = \Omega(w_{\mathrm{RES}}(\varphi)^2)$

COROLLARY. $\exists$ $O(1)$-CNF φ OF SIZE $O(n)$

WITH $S_{\mathrm{RES}}(\varphi) = \Omega(n^2)$

PROOF IDEA:
- INDUCTION ON w
- FIND A RESTRICTION ρ THAT
  - DECREASES $w_{\mathrm{RES}}(\varphi)$ BY AT MOST 1
  - SATISFIES $\geq w$ NODES IN THE REFUTATION

MAIN RESULT PROOF IDEA

RES($\oplus$) REFUTATION AS PARITY DECISION DAG:

- EACH NODE v IS LABELED BY $\mathbb{F}_2$-LINEAR SYSTEM Φ_v
- SOURCE IS LABELED BY EMPTY SYSTEM
- FOR SINK v $\exists$ CLAUSE C S.T. Φ_v IS INCONSISTENT WITH C
- EACH NON-SINK v
    - IS LABELED BY LINEAR FORM f_v
    - HAS TWO CHILDREN v_0, v_1
    - $\forall a \in \{0,1\}$ $\Phi_v \wedge (f_v = a)$ IMPLIES Φ_{v_a}

PROOF IDEAS

RES(⊕) REFUTATION AS PARITY DECISION DAG

- EACH NON-SINK v
  - IS LABELED BY LINEAR FORM f_v
  - HAS TWO CHILDREN v_0, v_1
  - $\forall a \in \{0,1\}$ $\varphi_v \wedge (f_v = a)$ IMPLIES φ_{v_a}

DAG

$$\begin{tikzcd}
 & \varphi_v \arrow[dl, "f_v=0"'] \arrow[dr, "f_v=1"] & \\
\varphi_{v_0} & & \varphi_{v_1}
\end{tikzcd}$$

RES(⊕) REFUTATION

$$\begin{tikzcd}
 & \neg\varphi_v & \\
\neg\varphi_v \vee \neg(f_v=0) \arrow[ur] & & \neg\varphi_v \vee \neg(f_v=1) \arrow[ul] \\
\neg\varphi_{v_0} \arrow[u] & & \neg\varphi_{v_1} \arrow[u]
\end{tikzcd}$$

MAIN RESULT PROOF IDEA

<u>THEOREM.</u> IF g IS 1-STIFLING, THEN $\forall \varphi$ $\mathrm{RES}(\oplus)$ ANY REFUTATION OF $\varphi \circ g$ OF SIZE S HAS DEPTH $\Omega(w_{RES}(\varphi)^2 / \log S)$

<u>PROOF IDEA:</u>

- INDUCTION ON w
- CONSIDER DAG G FOR $\varphi \circ g$ OF SIZE S
- FIND v IN G WITH DEPTH $v \geq \Omega(w)$
- FIX SOME VARIABLES IN A WAY THAT:
  - $G_{ROOT \to v}$ REFUTES SOME $\varphi' \circ g$
  - $w_{RES}(\varphi') \geq w_{RES}(\varphi) - O(\log S)$

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  - $G_{ROOT \to v}$ REFUTES SOME $\varphi' \circ g$
  - $w_{RES}(\varphi') \geq w_{RES}(\varphi) - O(\log S)$
- HOW TO FIND v?

MAIN RESULT PROOF IDEA

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<u>PROOF IDEA:</u>

- HOW TO FIND v?

MAIN RESULT PROOF IDEA

<u>THEOREM.</u> IF g IS 1-STIFLING, THEN $\forall \varphi$ RES($\oplus$) ANY REFUTATION OF $\varphi \circ g$ OF SIZE S HAS DEPTH $\Omega(w_{RES}(\varphi)^2 / \log S)$

<u>PROOF IDEA:</u>

- HOW TO FIND v?
- RANDOM WALK WITH RESTART FOR $O(w)$ STEPS
- FOR ANY v IF $\mathrm{RANK}(\Phi_v) = d \Rightarrow$

$$\Pr[\text{FINISH WALK IN } v] = 2^{-d}$$

- THEN $\exists$ PATH $w \to v$ WITH $\mathrm{RANK}(\Phi_v) = O(\log S)$
- FIX $O(\log S)$ INPUTS OF φ TO RESTART IN v

MAIN RESULT PROOF IDEA

<u>THEOREM.</u> IF g IS 1-STIFLING, THEN $\forall \varphi$ RES($\oplus$) ANY REFUTATION OF $\varphi \circ g$ OF SIZE S HAS DEPTH $\Omega(w_{RES}(\varphi)^2 / \log S)$

<u>PROOF IDEA:</u>

MAIN NOVELTY:

- ON RANDOM STEP WITH PROBABILITY $\frac{1}{2}$:
    - EITHER FIX A BLOCK OF VARIABLES
    - OR MAKE A RANDOM STEP IN DAG
- FIRST HELPS TO ENSURE LOW PROBABILITY TO REACH GIVEN v
- SECOND ENSURES LONG ENOUGH WALK

OPEN QUESTIONS

- STRONG LOWER BOUND BEYOND DEPTH-n^2 RES($\oplus$)

- SUPERQUADRATIC SIZE LOWER BOUND FOR RES($\oplus$)

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- STRONG LOWER BOUND BEYOND DEPTH-n^2 RES($\oplus$)
- SUPERQUADRATIC SIZE LOWER BOUND FOR RES($\oplus$)

THANK YOU!