Almost Optimal Distribution-free Junta Testing

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Technion

k-Junta

A Boolean function $f:\{0,1\}^n \to \{0,1\}$ is called k −junta if it depends on at most k variables/coordinates.

Examples:

$$f(x_1, x_2, \dots, x_{10000}) = x_{1111} \wedge (x_{9992} \oplus x_{3001})$$

is 3-junta

is 4-junta

Relevant variables/coordinates

1-junta is $\{x_i, \overline{x_i}, 0,1\}$ 0-junta is $\{0,1\}$

k −Junta is the class of all k −juntas.

Model of Testing

Given a black box that contains a Boolean function $f: \{0,1\}^n \rightarrow \{0,1\}$

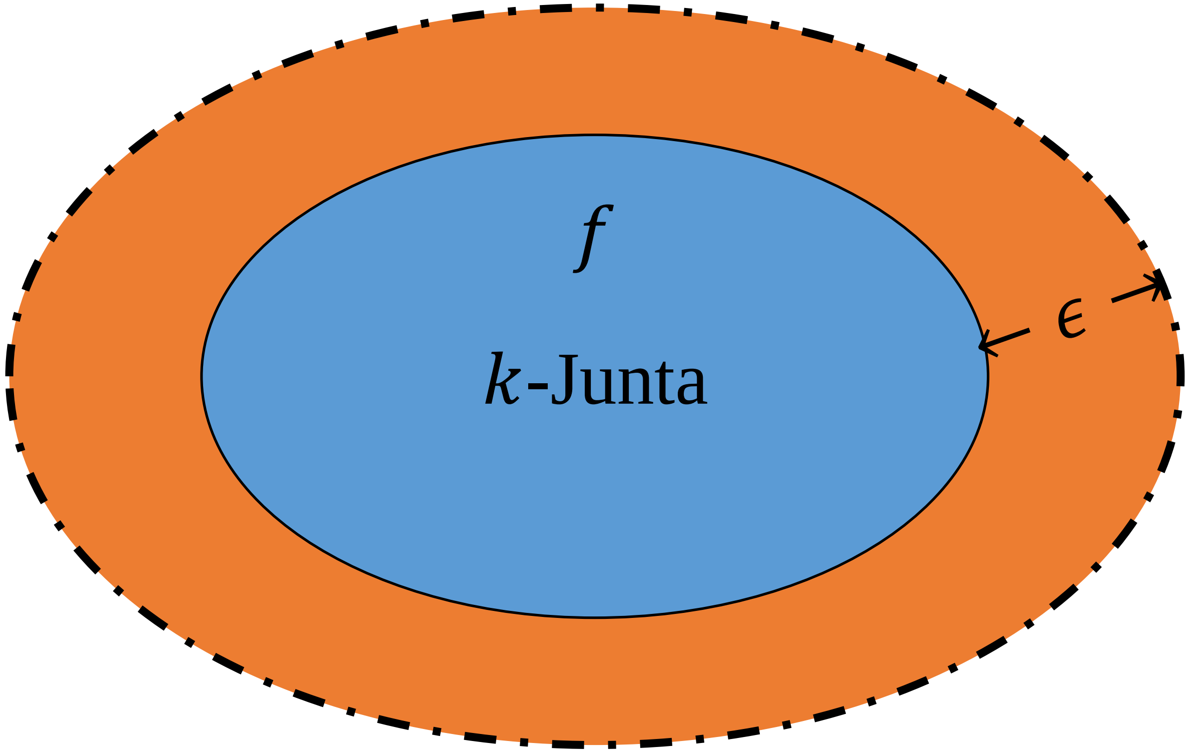
Given two oracles: 1) when $x \in \{0,1\}^n$ is queried, it returns $f(x)$
and 2) returns $u \in \{0,1\}^n$ chosen acc. to unknown distribution $\mathcal{D}$

A *distribution-free testing algorithm* A for k −Junta is an algorithm that, given an access to the two oracles and a distance parameter ϵ as an input ,

1) if f is k-junta then A outputs "accept" with probability at least 2/3.
2) if f is ϵ-far from every k-junta with respect to the distribution $\mathcal{D}$ then A outputs "reject" with probability at least 2/3.

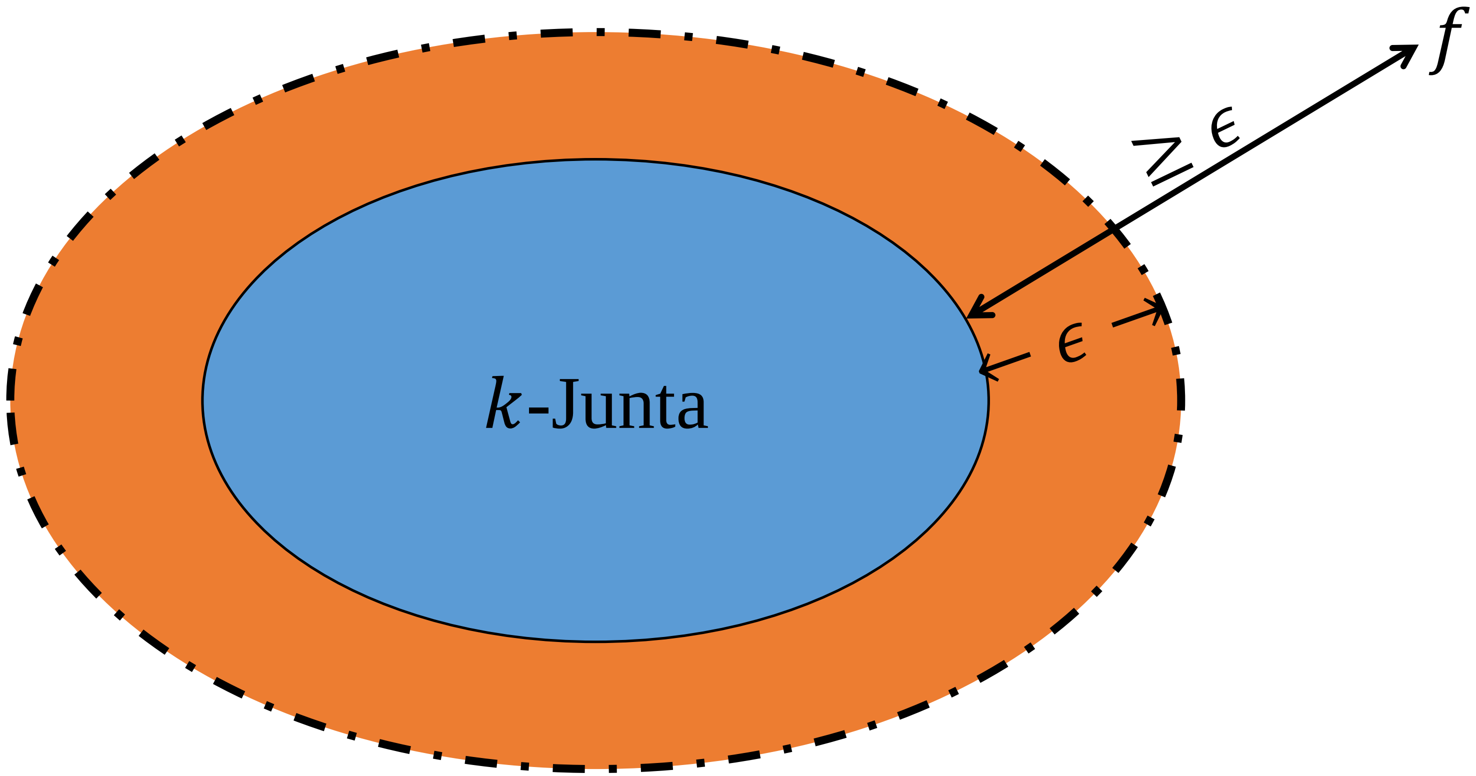
$$(\forall\, h \in k - Junta) \quad \Pr_{x\sim\mathcal{D}}[f(x) \neq h(x)] \geq \epsilon$$

Model of Testing



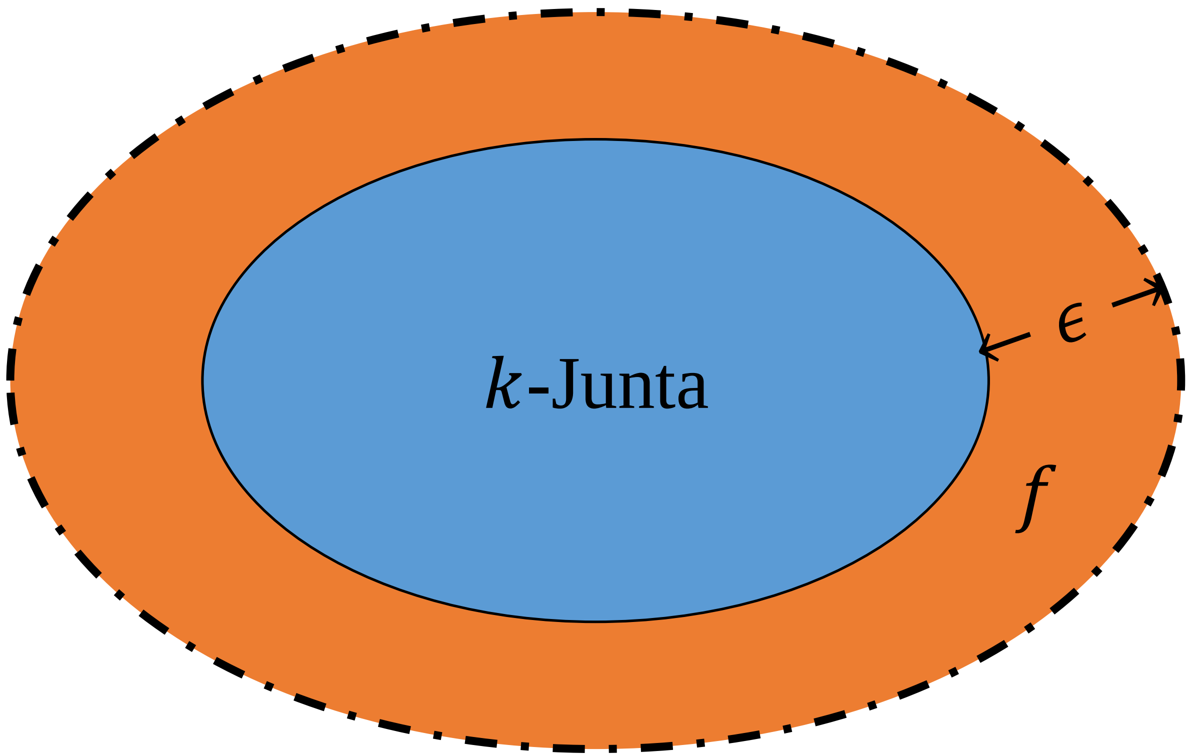
A outputs "accept" with probability at least 2/3.

Model of Testing



A outputs “reject” with probability at least 2/3.

Model of Testing



A halts and outputs either "accept" or "reject".

Results in the uniform distribution Model

Reference	Ada./NonAda	Lo./Up.	Result $\tilde{O}/\tilde{\Omega}$

Adaptive

NonAdaptive

Lower bounds

Upper bounds

For the number of queries

$$\tilde{O}(T) = T\, poly(\log T)$$

$$\tilde{\Omega}(T) = T\, / poly(\log T)$$

Time $poly\left(n, \frac{1}{\epsilon}\right)$

Results in the uniform distribution Model

Reference	Ada./NonAda	Lo./Up.	Result $\widetilde{O}/\widetilde{\Omega}$
Blais STOC 2009	Adaptive	Upper	k/ϵ

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Chen et al. CCC 2017	NonAdaptive	Lower	$k^{\frac{3}{2}}/\epsilon$

Results in the distribution-free model

Reference	**Ada./NonAda**	**Lo./Up.**	**Result $\widetilde{O}/\widetilde{\Omega}$**

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Halevy et al. APPROX 03	NonAdaptive	Upper	$2^k/\epsilon$

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Ours	Adaptive	Upper	k/ϵ

<u>The algorithm</u>

$$f(x_1, \ldots, x_n)$$

Choose a random uniform partition $X_1, X_2, \ldots, X_r$ of $[n]$ where $r = 2k^2$

$$f(x_{X_1} \circ x_{X_2} \circ \cdots \circ x_{X_r})$$

Why $2k^2$? If f is k−junta, whp each X_i contains at most one relevant coordinate

<u>Find relevant sets</u>

Find <u>relevant sets</u> $X_{i_1}, X_{i_2}, \ldots, X_{i_{k'}}$ $f(v) \neq f\left(v_{X_i} \circ 0_{X_i}\right)$

$X_{i_1}, X_{i_2}, \ldots, X_{i_j}$

$X = X_{i_1} \cup X_{i_2} \cup \cdots \cup X_{i_j}$ $f(u_X \circ 0_{\bar{X}}) ?= f(u)$ $u \sim \mathcal{D}$

Find relevant sets

Find relevant sets $X_{i_1}, X_{i_2}, \ldots, X_{i_{k'}}$

$$X_{i_1}, X_{i_2}, \ldots, X_{i_j}$$

$X = X_{i_1} \cup X_{i_2} \cup \cdots \cup X_{i_j}$ $\quad$ $f(u_X \circ 0_{\bar{X}}) \mathrel{?=} f(u)$ $\quad$ $u \sim \mathcal{D}$

$f(u_X \circ 0_{\bar{X}}) \neq f(u)$ $\quad$ Find a new relevant set $X_{i_{j+1}} = X_\ell$

$\not\approx$ $\quad$ $\not\approx$

$$f\left(u_X \circ u_{Y_1} \circ 0_{Y_2}\right)$$

$$\bar{X} = Y_1 \cup Y_2$$

Find relevant sets

Find relevant sets $X_{i_1}, X_{i_2}, \dots, X_{i_{k'}}$

$X_{i_1}, X_{i_2}, \dots, X_{i_j}$

$X = X_{i_1} \cup X_{i_2} \cup \cdots \cup X_{i_j}$ $\quad f(u_X \circ 0_{\bar{X}}) \mathrel{?}= f(u)$ $\quad u \sim \mathcal{D}$

$f(u_X \circ 0_{\bar{X}}) \neq f(u)$ Find a new relevant sets $X_{i_{j+1}} = X_\ell$ $\quad \log r = O(\log k)$

If this is the $(k+1)$-th relevant set then "reject"

We also get a witness for X_ℓ

$$f\left(v^{(\ell)}\right) \neq f\left(v^{(\ell)}_{\overline{X_\ell}} \circ 0_{X_\ell}\right)$$

$f(u_X \circ 0_{\bar{X}}) = f(u)$ $\quad$ For $\tilde{O}\left(\frac{1}{\epsilon}\right)$ values of $u \sim \mathcal{D}$

$$\Pr_{x \sim \mathcal{D}}\left[f(x_X \circ 0_{\bar{X}}) \neq f(x)\right] \leq \epsilon/3$$

$\tilde{O}\left(\frac{k}{\epsilon}\right)$ queries

The algorithm

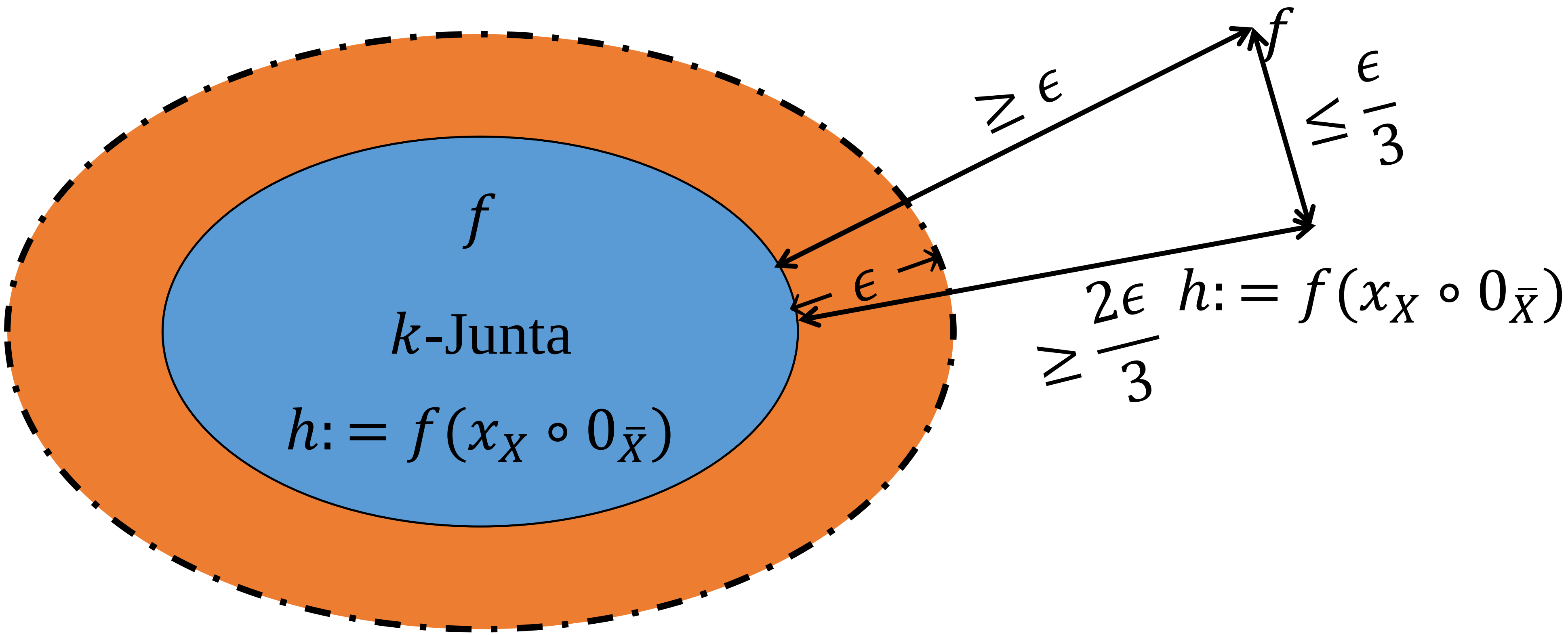
$X_{i_1}, X_{i_2}, \ldots, X_{i_{k'}}$ $, k' \leq k$

Each X_{i_j} contains at least one relevant variable

$X = X_{i_1} \cup X_{i_2} \cup \cdots \cup X_{i_{k'}}$

$$\Pr_{x \sim \mathcal{D}} [f(x_X \circ 0_{\bar{X}}) \neq f(x)] \leq \frac{\epsilon}{3}$$

$h := f(x_X \circ 0_{\bar{X}})$ is $\frac{\epsilon}{3}$−close to f with respect to $\mathcal{D}$



$X_{i_1}, X_{i_2}, \ldots, X_{i_{k'}}\ \ , k' \leq k$ Each X_{i_j} contains at least one relevant coordinte

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$\tilde{O}\left(\frac{k}{\epsilon}\right)$ queries

f is k −junta

$h := f(x_X \circ 0_{\bar{X}})$ is k −junta

Whp each X_{i_j} contains exactly one relevant coordinate

$f(x)$ is ϵ −far from every k −Junta with respect to $\mathcal{D}$

$h := f(x_X \circ 0_{\bar{X}})$ is $\frac{2\epsilon}{3}$ −far from every k −Junta with respect to $\mathcal{D}$

We also have witness for each relevant set X_{i_j} – that is,

$$f\left(v^{(j)}\right) \neq f\left(v^{(j)}_{\overline{X_{i_j}}} \circ 0_{X_{i_j}}\right)$$

f is k −junta

$f(x)$ is ϵ −far from every k −Junta with respect to $\mathcal{D}$

$h := f(x_X \circ 0_{\bar{X}})$ is k −junta

Whp each X_{i_j} contains exactly one relevant variable

$h := f(x_X \circ 0_{\bar{X}})$ is $\frac{2\epsilon}{3}$ −far from every k −Junta with respect to $\mathcal{D}$

$$f\left(v^{(j)}\right) \neq f\left(v^{(j)}_{\overline{X_{i_j}}} \circ 0_{X_{i_j}}\right)$$

$f\left(v^{(j)}_{\overline{X_{i_j}}} \circ x_{X_{i_j}}\right)$ is equal to $x_{\tau(j)}$ or $\overline{x_{\tau(j)}}$

$\tilde{O}(k)$ queries

$O(\log n)$

1 −Junta\0 −Junta

$\tilde{O}(1)$ queries

$f\left(v^{(j)}_{\overline{X_{i_j}}} \circ x_{X_{i_j}}\right)$ is (1/15)-close to a literal in $\{x_{\tau(j)}, \overline{x_{\tau(j)}}\}$ according to the uniform distribution

$h := f(x_X \circ 0_{\bar{X}})$ is either k −junta that depends on $\Gamma = \{\tau(1), \tau(2), \dots, \tau(k')\}$

Or $\frac{2\epsilon}{3}$ −far from every k −junta w.r.t. $\mathcal{D}$ $X_{i_1}, X_{i_2}, \dots, X_{i_{k'}}$

Fix any $y \in \{0,1\}^n$ $g = h(x_\Gamma \circ y_{\bar{\Gamma}})$ is k-junta

$h(x)$ is k −junta

$h(x)$ is $\frac{2\epsilon}{3}$ −far from any k-junta with respect to $\mathcal{D}$

$$h(x) = h(x_\Gamma \circ y_{\bar{\Gamma}})$$

$$\Pr_{u \sim \mathcal{D}}[h(u) \neq h(u_\Gamma \circ y_{\bar{\Gamma}})] \geq \frac{2\epsilon}{3}$$

$$\Pr_{u \sim \mathcal{D}, y \sim U}[h(u) \neq h(y) | y_\Gamma = u_\Gamma] = 0$$

$$\Pr_{u \sim \mathcal{D}, y \sim U}[h(u) \neq h(u_\Gamma \circ y_{\bar{\Gamma}})] \geq \frac{2\epsilon}{3}$$

$\tilde{O}\left(\frac{k}{\epsilon}\right)$ queries

$\tilde{O}\left(\frac{1}{\epsilon}\right)$ queries

$$\Pr_{u \sim \mathcal{D}, y \sim U}[h(u) \neq h(y) | y_\Gamma = u_\Gamma] \geq \frac{2\epsilon}{3}$$

Given u?

How to draw a random uniform y such that $y_\Gamma = u_\Gamma$? $\tilde{O}(k)$ queries

$\Gamma = \{\tau(1), \tau(2), \dots, \tau(k')\}$ $\qquad$ $h := f(x_X \circ 0_{\bar{X}})$

Given u?

How to draw a random uniform y such that $y_\Gamma = u_\Gamma$?

$f\left(v^{(j)}_{\overline{X_{i_j}}} \circ x_{X_{i_j}}\right)$ is (1/15)-close to a literal in $\{x_{\tau(j)}, \overline{x_{\tau(j)}}\}$ wrt uniform dist.

↳A procedure that given u finds $u_{\tau(j)}$ with high probability

$\tilde{O}(1)$ queries

$$y = y_{X_{i_1}} \circ y_{X_{i_2}} \circ \cdots \circ y_{X_{i_{k'}}} \circ y_{\bar{X}}$$

Draw a random uniform $y_{X_{i_j}}$

If $y_{\tau(j)} = u_{\tau(j)}$ then output($y_{X_{i_j}}$)

If $y_{\tau(j)} \neq u_{\tau(j)}$ then output($\overline{y_{X_{i_j}}}$)

$\tilde{O}(k)$ queries

Chen, Liu, Servedio, Sheng, Xie 2018

$\blacksquare$

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	$O(1)$-Round		?
Ours CCC 2019	Adaptive	Upper	k/ϵ
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Almost non-Adaptive

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	$O(1)$-Round		?
	?-Round		$\mathrm{poly}(k/\epsilon)$
Ours CCC 2019	Adaptive	Upper	k/ϵ
Saglam FOCS 2018	Adaptive	Lower	k

Thank You