Optimality of Linear Sketching under Modular Updates

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Streaming and sketching

Streaming with binary updates

- Counters $x_1, \ldots, x_n \in \mathbb{F}_2$
- Stream of updates: $x_i \leftarrow x_i \oplus 1$
- At the end, want to compute function $f(x_1, \ldots, x_n)$
- For which functions can we do it using $\ll n$ memory?

Example

- Initially 000000
- Flip x_1 100000
- Flip x_5 100010
- Flip x_2 110010
- Flip x_5 110000
- …

- Compute $f(x_1, \dots, x_n)$

Linear sketching

- Linear sketching is a useful primitive for streaming

- Let $f: \mathbb{F}_2^n \to \{0,1\}$

- f has a linear sketch of size k if it factors as $f(x) = p(L(x))$ where:

(i) $L: \mathbb{F}_2^n \to \mathbb{F}_2^k$ linear function

(ii) $p: \mathbb{F}_2^k \to \{0,1\}$ post-processing function

- Equivalently, the “Fourier dimension” of f is k

Linear sketching implies streaming

- Assume $f: \mathbb{F}_2^n \to \{0,1\}$ factors as $f(x) = p(L(x))$ where

(i) $L: \mathbb{F}_2^n \to \mathbb{F}_2^k$ linear function

(ii) $p: \mathbb{F}_2^k \to \{0,1\}$ post-processing function

- To compute f in the streaming model, maintain $L(x) \in \mathbb{F}_2^k$
- Easy to maintain under updates $x_i \leftarrow x_i \oplus 1$
- Requires only k bits of memory

Randomized linear sketching

- Randomization makes linear sketching more powerful

- $f: \mathbb{F}_2^n \to \{0,1\}$ has a randomized linear sketch of size k if it can be approximated by a distribution over linear sketches of size k

- That is, if exists a distribution over (L, p), where:

(i) $L: \mathbb{F}_2^n \to \mathbb{F}_2^k$ linear function

(ii) $p: \mathbb{F}_2^k \to \{0,1\}$ post-processing function

Such that $\Pr_{L,p}[f(x) = p(L(x))] \geq 1 - \epsilon$

Randomized sketching gives additional power

- Consider the OR function: $OR(x_1, \dots, x_n) = x_1 \vee \cdots \vee x_n$

- Deterministic sketching requires size n

- Randomized sketching can be done in size $O(\log(1/\epsilon))$

(random parities)

Is linear sketching universal?

- Linear sketching seems like a very useful primitive for streaming
- Is it universal?

- That is: given a streaming algorithm that computes f using k bits of memory, can we extract from it a linear sketch for f of size $\approx k$?

Universality of linear sketching

Universality of linear sketching

- Let $f: \mathbb{F}_2^n \to \{0,1\}$
- Assume: randomized streaming algorithm supporting N updates and using k bits of memory
- Goal: extract a randomized linear sketch of size $\approx k$

- True if $N \geq 2^{2^{2^n}}$ [Li-Nguyen-Woordruff '14, Ai-Hu-Li-Woodruff '16]

- True if $N = \Omega(n)$ for random inputs [Kannan-Mossell-Sanyal-Yaroslavtsev '18]

- True if $N = \Omega(n^2)$ [This work]

Main theorem: streaming

- Let $f: \mathbb{F}_2^n \to \{0,1\}$

- Assume there exists a randomized streaming algorithm for f supporting $\mathrm{N} = \Omega(n^2)$ updates which uses k bits of memory

- Then there exists a randomized linear sketch for f of size $O(k)$

Extensions (that I will not talk about)

- Extends to approximate real-valued functions $f: \mathbb{F}_2^n \to [0,1]$
- Extends to functions over other fields
- Assuming only $\mathrm{N} = \Omega(n)$ updates are supported, we can still extract a randomized linear sketch, but its size will be $poly(k)$ instead of $O(k)$

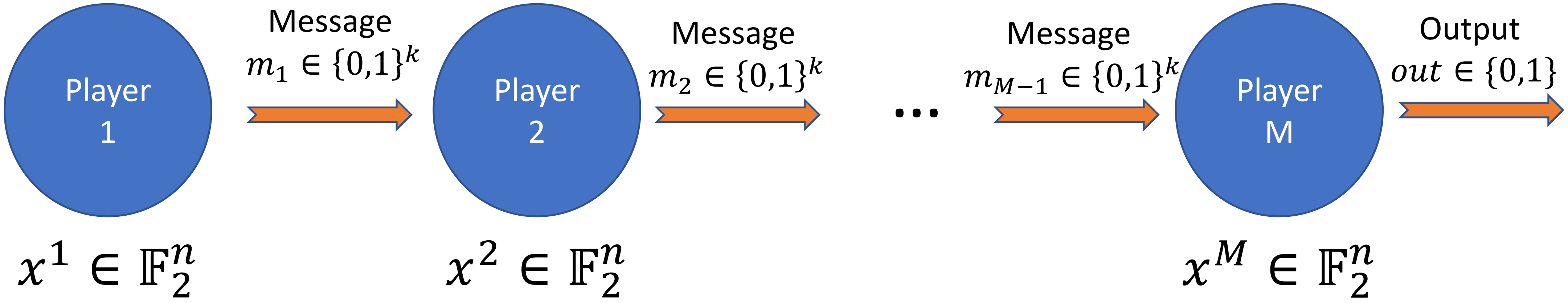
One-way communication complexity

One way communication complexity

- Model a streaming algorithm as a one-way communication protocol
- Break N updates into $M = N/n$ chunks of size n each
- Setup: M players, holding inputs $x^1, \ldots, x^M \in \mathbb{F}_2^n$
(x^i is the aggregate of the n updates in the i-th chunk)
- Goal: compute $f\left(x^1 + \cdots + x^M\right)$
- Communication model: one-way

One way communication complexity

- M players, holding inputs $x^1, \ldots, x^M \in \mathbb{F}_2^n$
- Model: one-way communication with shared randomness
- Goal: $\text{output} = f\left(x^1 + \cdots + x^M\right)$ w.h.p over shared randomness



Main theorem: one way communication

- Let $f: \mathbb{F}_2^n \to \{0,1\}$
- Assume there exists a one-way communication protocol for computing $f\left(x^1 + \cdots + x^M\right)$ for $M = \Omega(n)$ players with k-bit messages

(recall: this corresponds to $N = Mn = \Omega\left(n^2\right)$ binary updates)

- Then there exists a randomized linear sketch for f of size $O(k)$
- For $M = \Omega(1)$ players, get linear sketch of size $poly(k)$

Proof

Proof

- The proof uses

1. Standard techniques in communication complexity
2. Additive combinatorics

Proof step 1: Yao's minimax principle

- Let $f: \mathbb{F}_2^n \to \{0,1\}$
- Fix a "hard distribution" μ over inputs
- Goal: linear sketch for $f(x)$ where $x \sim \mu$

- Embed hard distribution to the M players:
- First M-1 players inputs $x^1, \ldots, x^{M-1}$ are uniform in $\mathbb{F}_2^n$
- Last player input x^M is set so that $x^1 + \cdots + x^M = x$

- Intuition: protocol has no information on x until the last player

Proof step 2: protocol structure

- Target: $x \sim \mu$
- Players inputs: $x^1, \dots, x^{M-1} \in \mathbb{F}_2^n$ uniformly, $x^M = x^1 + \cdots + x^{M-1} + x$

- We may assume the protocol is deterministic
- Messages: $m_1(x^1), m_2(m_1, x^2), m_3(m_1, m_2, x^3), \dots$
- Output: $out(m_1, \dots, m_{M-1}, x^M)$

- With good probability $\text{out} = f(x^1 + \cdots + x^M) = f(x)$

- Can fix the messages (of the first M-1 players) to “typical messages”, without hurting the success probability too much

Proof step 3: fixing to typical messages

- Fix typical messages $m_1^*, m_2^*, \dots, m_{M-1}^*$

- Corresponds to the first M-1 players inputs:
  - $A_1 = \{x^1 \in \mathbb{F}_2^n : m_1(x^1) = m_1^*\}$
  - $A_2 = \{x^2 \in \mathbb{F}_2^n : m_2(m_1^*, x^2) = m_2^*\}$
  - …

- Sets are big: if the protocol uses k bits, then $|A_i| \geq 2^{n-k}$

- After conditioning on $x^1 \in A_1, \dots, x^{M-1} \in A_{M-1}$, protocol output is a function of only $x^M = x^1 + \cdots + x^{M-1} + x$

Proof step 4: mixing

- Large sets $A^1, \ldots, A^{M-1} \subset \mathbb{F}_2^n$ of density 2^{-k}
- If we sample $x^1 \in A_1, \ldots, x^{M-1} \in A_{M-1}$ and $x \sim \mu$, then with high probability

$$out\left(x^1 + \cdots + x^{M-1} + x\right) = f(x)$$

- Technical lemma: for $M = \Omega(N)$, the sum $x^1 + \cdots + x^{M-1}$ mixes in $\mathbb{F}_2^n$
- More precisely, there exists a subspace $V \subset \mathbb{F}_2^n$ of co-dimension $O(k)$, such that the sum is near invariant to a random shift from V

Proof step 5: extracting linear sketch

- We found a large subspace V of co-dimension O(k)
- If we sample $x^1 \in A_1, \ldots, x^{M-1} \in A_{M-1}$, $x \sim \mu$ and $v \in V$, then with high probability

$$out\left(x^1 + \cdots + x^{M-1} + x + v\right) = f(x)$$

- This allows to “factor out” V from the output function, and extract a linear sketch for $f(x)$

Open problems

Linear sketching for modular updates

- For binary updates (or more general, modular updates), we prove that linear sketching is universal
- Any streaming algorithm which supports $N = \Omega(n^2)$ updates implies a randomized linear sketch with similar guarantees
- **Open problem 1:** can this be improved to $N = \Omega(n)$?
- [Kannan-Mossell-Sanyal-Yaroslavtsev ‘18] proved a partial result in this regime, giving a linear sketch for f on random inputs
- Our results in this regime incur a polynomial loss in the sketch size

Integer updates

- Streaming if often considered in the integer case
- Integer counters $x_1, \dots, x_n$
- Updates $x_i \mathrel{+}= 1$ or $x_i \mathrel{-}= 1$
- Sketching corresponds to linear functions over the integers

- The results of [Li-Nguyen-Woordruff '14, Ai-Hu-Li-Woodruff '16] work in this regime as well, but require assuming $N \geq 2^{2^{2^n}}$

- **<u>Open problem 2:</u>** can our techniques be imported to this regime?
- Challenge: not clear what "mixing" should mean here

Thank you!