Optimal Short-Circuit Resilient Formulas

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Motivation

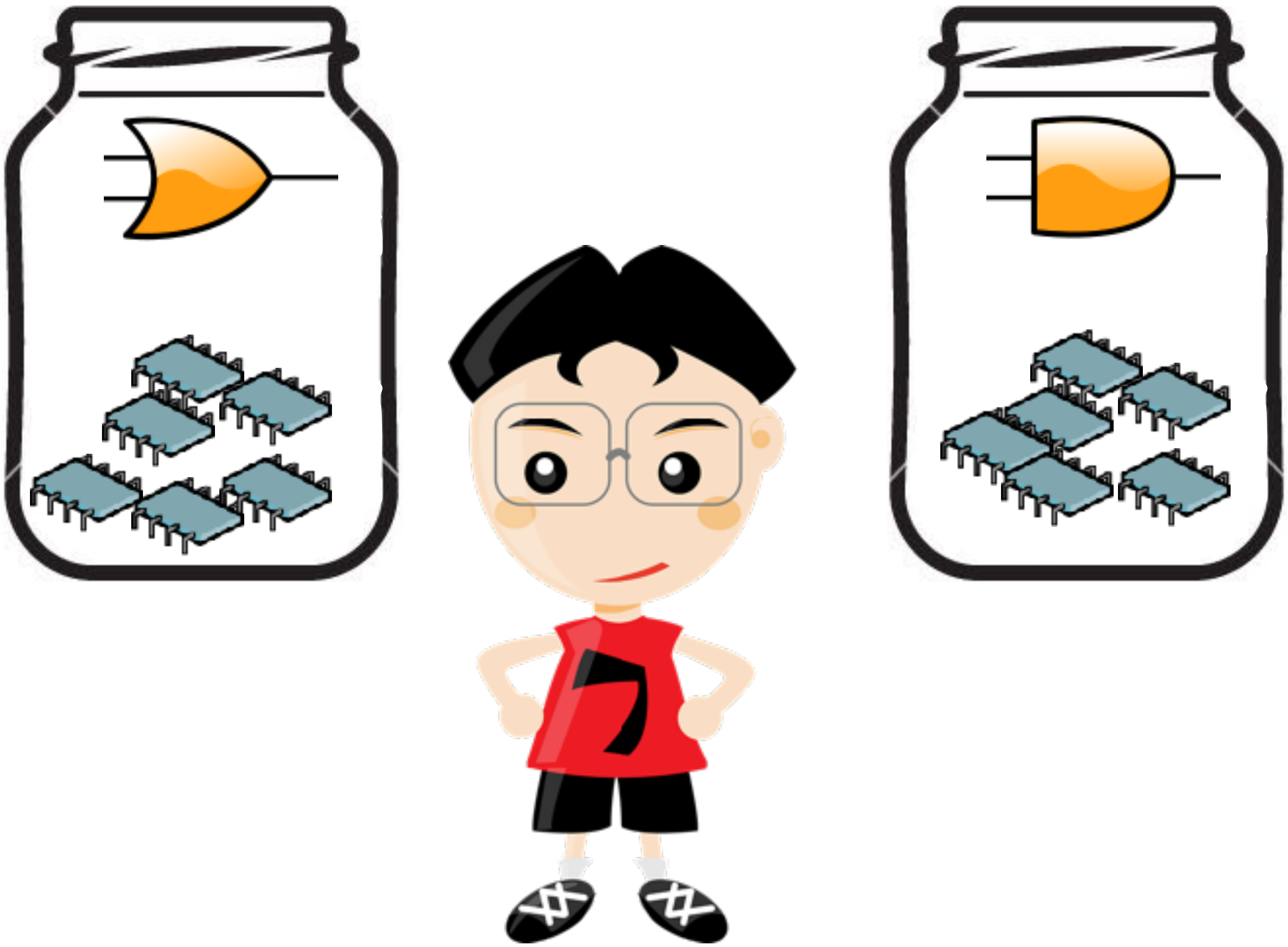
- How to construct a circuit that computes

$$f(z) = z_1 \wedge z_2$$

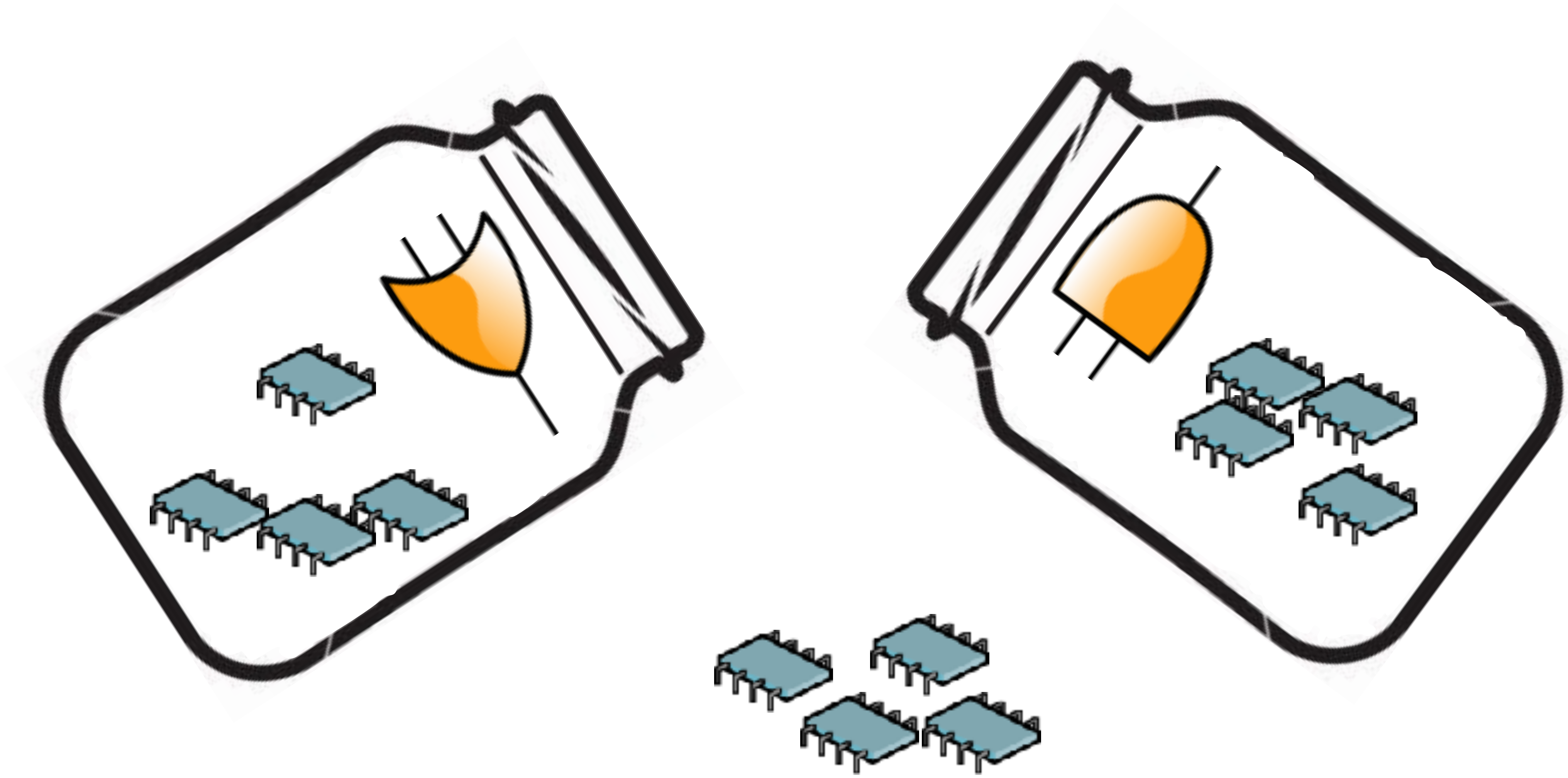
- Assuming AND / OR gates
  (all negations pushed to literals)

- EASY:

Motivation



Motivation



Motivation

- How to construct a circuit that computes

$$f(z) = z_1 \wedge z_2$$

- Assuming AND / OR   gates
- **When few of the AND / OR gates were mixed?**

Motivation

- A More General Question:
  - given a boolean function

$$f(z) : \{0,1\}^n \rightarrow \{0,1\}$$

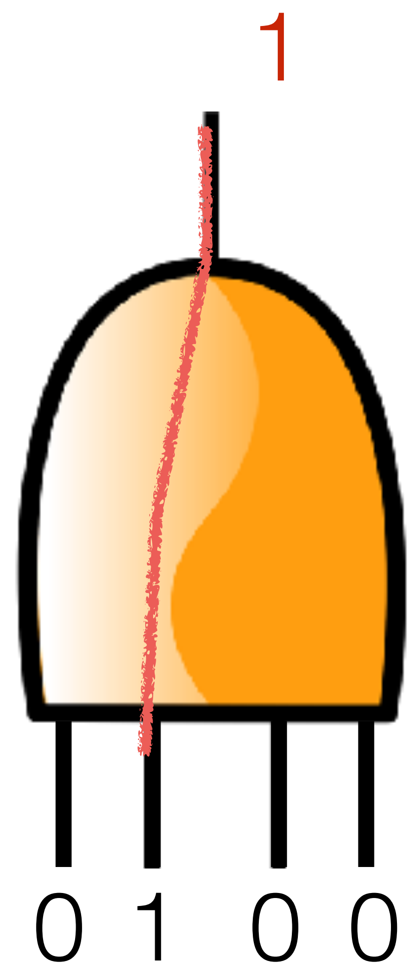
  - construct an AND / OR circuit for f, that works **even if a *constant fraction* of the gates are "faulty"**

Short-Circuit Noise

- A generalization of the above is a faulty gate with "short-circuit" noise

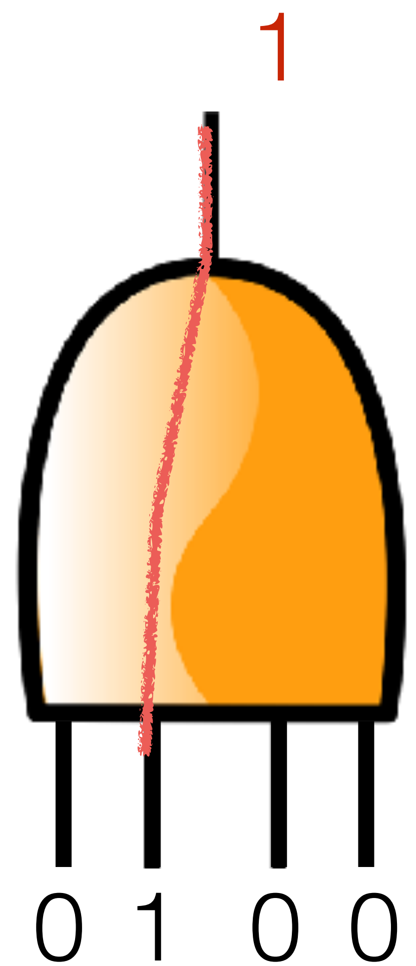
- The shorted input can be determined adversarially

- Equivalent to replacing the gate with an arbitrary gate g for which $g(0...0)=0$ and $g(1\ldots1)=1$



Short-Circuit Noise

- This type of noise is very common in produced wafers
- Incomparable to von-Neumann Noise (every wire flips w.p ε)



Short-Circuit Noise

- Main question(s):
  - How to construct an AND/OR circuit that is correct with up to k faulty (short-circuited) gates
  - What is the maximal k ?
    What is the maximal *fraction* of faulty gates?
  - How many extra gates we need to "fortify" a given circuit?

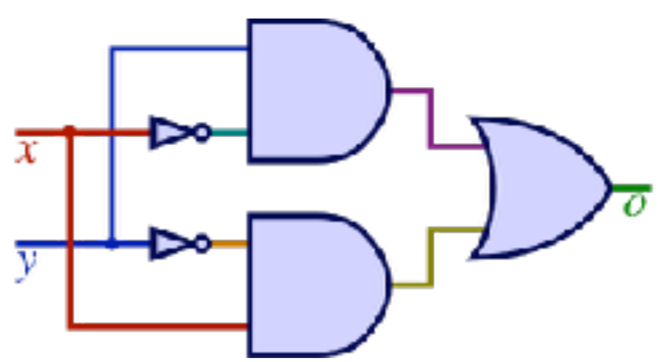
Prior Work

Work	Noise level	Circuit	Size
Kleitman-Leighton-Ma (J.Comp.Sys.Sci97)	k errors	any	$O(k\,\lvert C\rvert + k^{\log 3})$
Kalai-Lewko-Rao (FOCS12)	$\boldsymbol{\delta}$<1/6 fraction	formula (fan-in>2)	poly($\lvert F \rvert$)
	$\boldsymbol{\delta}$<1/10 fraction (*in-to-out path)	formula (fan-in=2)	

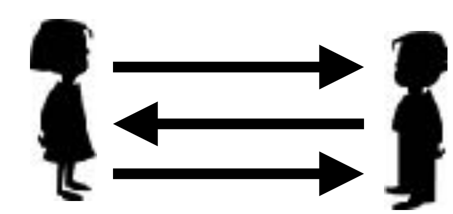
- Resilient Circuits with Von Neuman Noise:
  VonNeuman56, Dobrushin-Ortyukov77, Pippenger88, Pippenger89, Feder89, Gál91, Hajek-Weller91, Reischuk-Schmeltz91, Evans-Schulman99, Gács-Gál94, Evans-Pippenger98, Evans-Schulman03, Unger08/10, Mozeika-Saad-Raymond10

Resilient Formulas

- The Attack Plan: [Kalai-Lewko-Rao 2012]

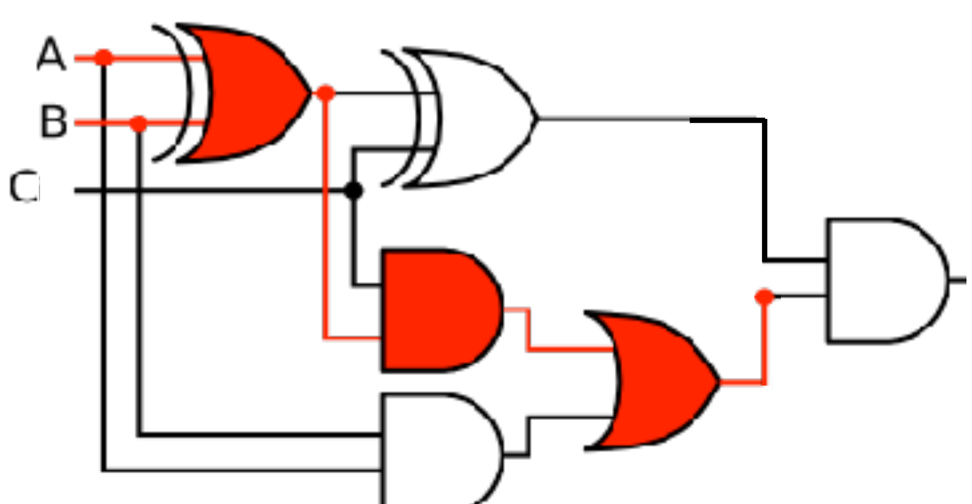


[KarchmerWigderson90]



δ-resilience

Coding w/
feedback [EGH16]
δ≤1/3



δ/2-resilience

Resilient Formulas

- Why do we lose a factor-2 in the resilience?
- Noise is one-sided:
  - Noise on AND gates can only make $0 \to 1$
  - Noise on OR gates can only make $1 \to 0$
- If out=1, noise on AND gates is meaningless!
- If a circuit is resilient to $\boldsymbol{\delta}$'-fraction, then

  (1) corrupting $\boldsymbol{\delta}$'-fraction of ANDs is OK, but also

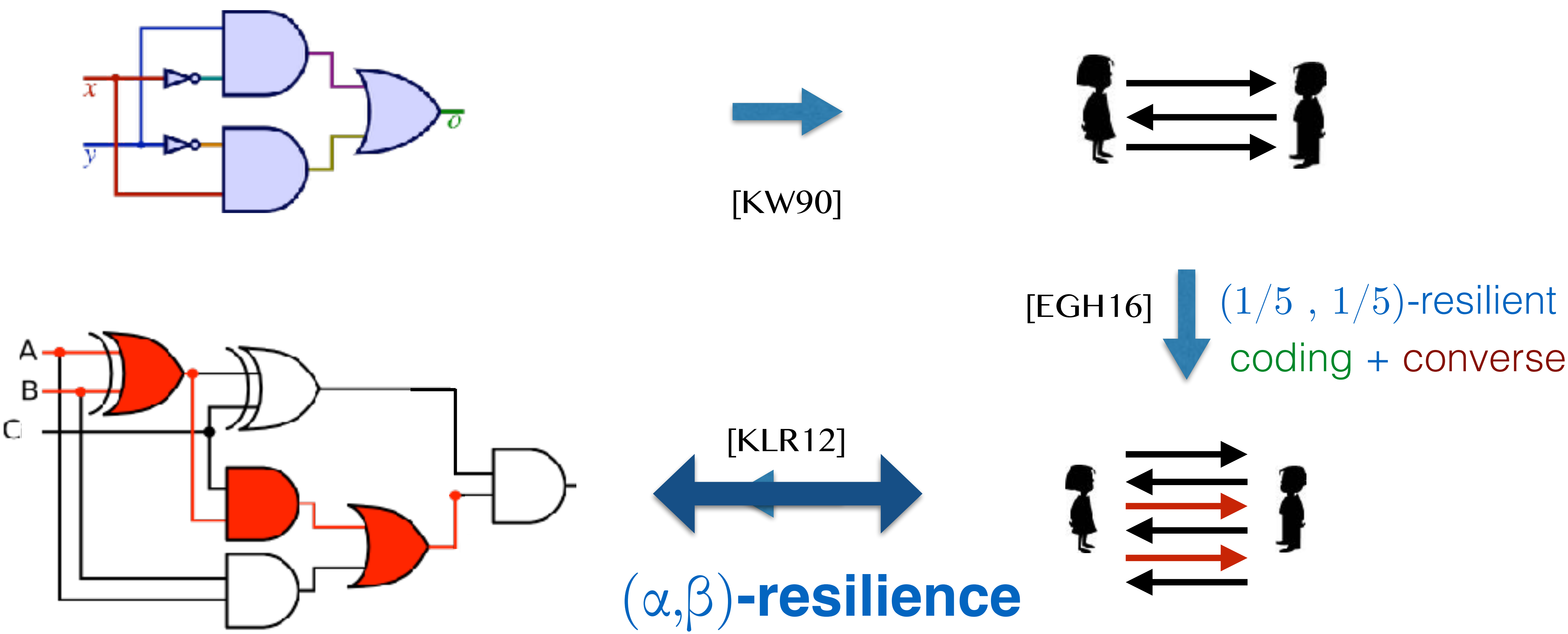
  (2) corrupting $\boldsymbol{\delta}$'-fraction of ORs is OK

  $\Rightarrow$ is resilient to $2\boldsymbol{\delta}$', thus $2\boldsymbol{\delta}' \leq \boldsymbol{\delta}$ (since res. comes from protocol)

Resilient Formulas

- Idea: split the noise to AND and OR gates
- Def. (α,β)**-corruption** means corrupting at most αn AND gates and βn OR gates in every in-to-out path (n is depth of circuit)

Result


x
y
o
[KW90]
[EGH16]
(1/5 , 1/5)-resilient
coding + converse
A
B
C
[KLR12]
(α,β)-resilience

Main Result

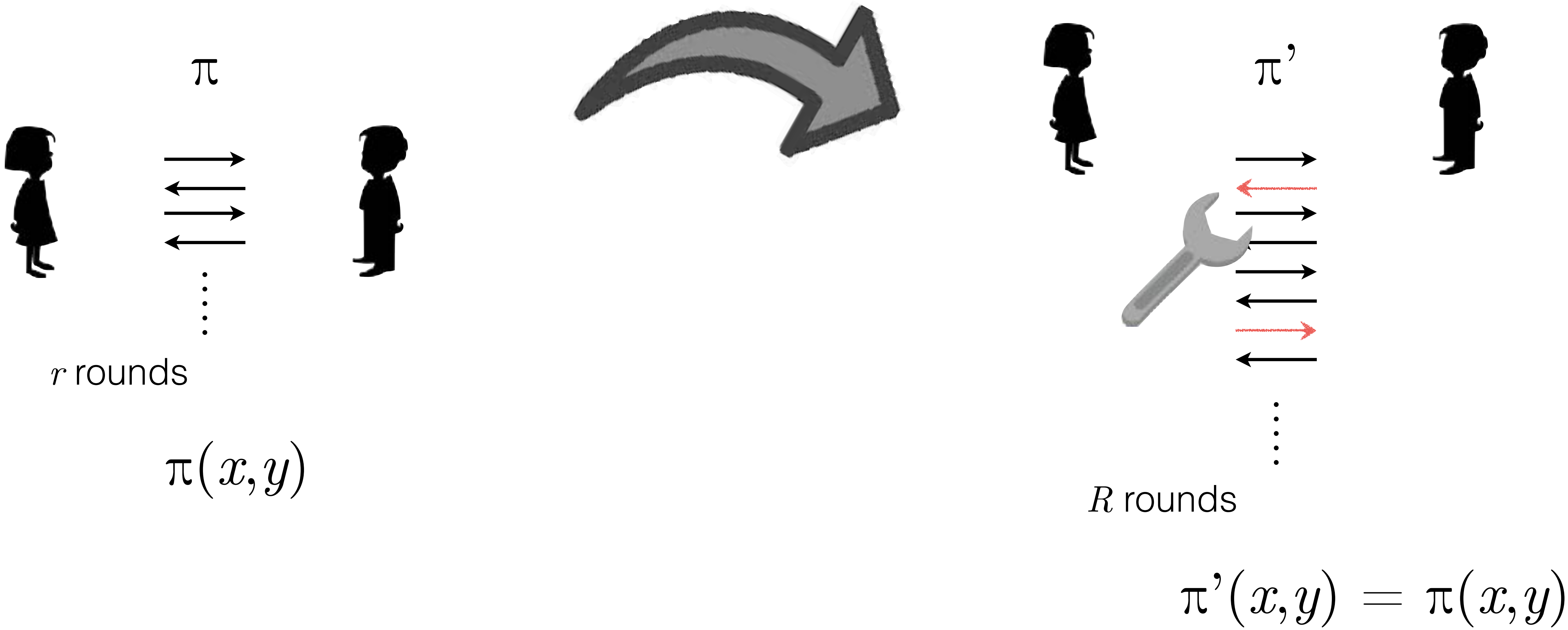
- **Upper Bound (Direct):**
  Any formula F can be (efficiently) compiled into F' so that:
  - F' is correct if up to
    1/5-ε fraction of the AND-gates, *and* **1/5-ε** of the OR-gates
    are faulty in *any* input-to-output branch
  - F' has constant fan in (> 2), $|F'| = \mathrm{poly}(|F|)$

- **Lower Bound (Converse) :** Resilience of 1/5 is ***tight***.
  There exist functions that 1/5 corruption invalidates any F of sub-exponential size

Techniques: Upper Bound

(1/5,1/5)-resilient coding scheme w/ feedback

Coding for Interactive Comm.

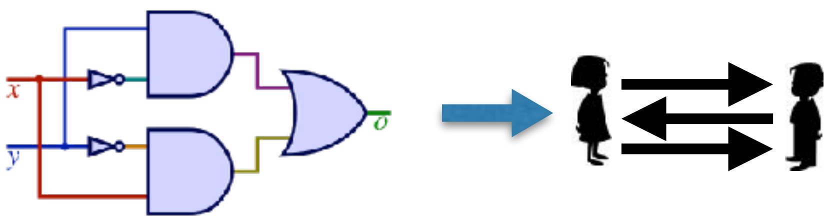


Many Coding Schemes exist for various settings

[Schulman96, GMS14, BR14, KR13, GH15, Pan13, EGH16, Hau14, BK12, BKN14, FGOS15, BGMO16, BNTTU14, G17, GHKZW18] …

Feedback

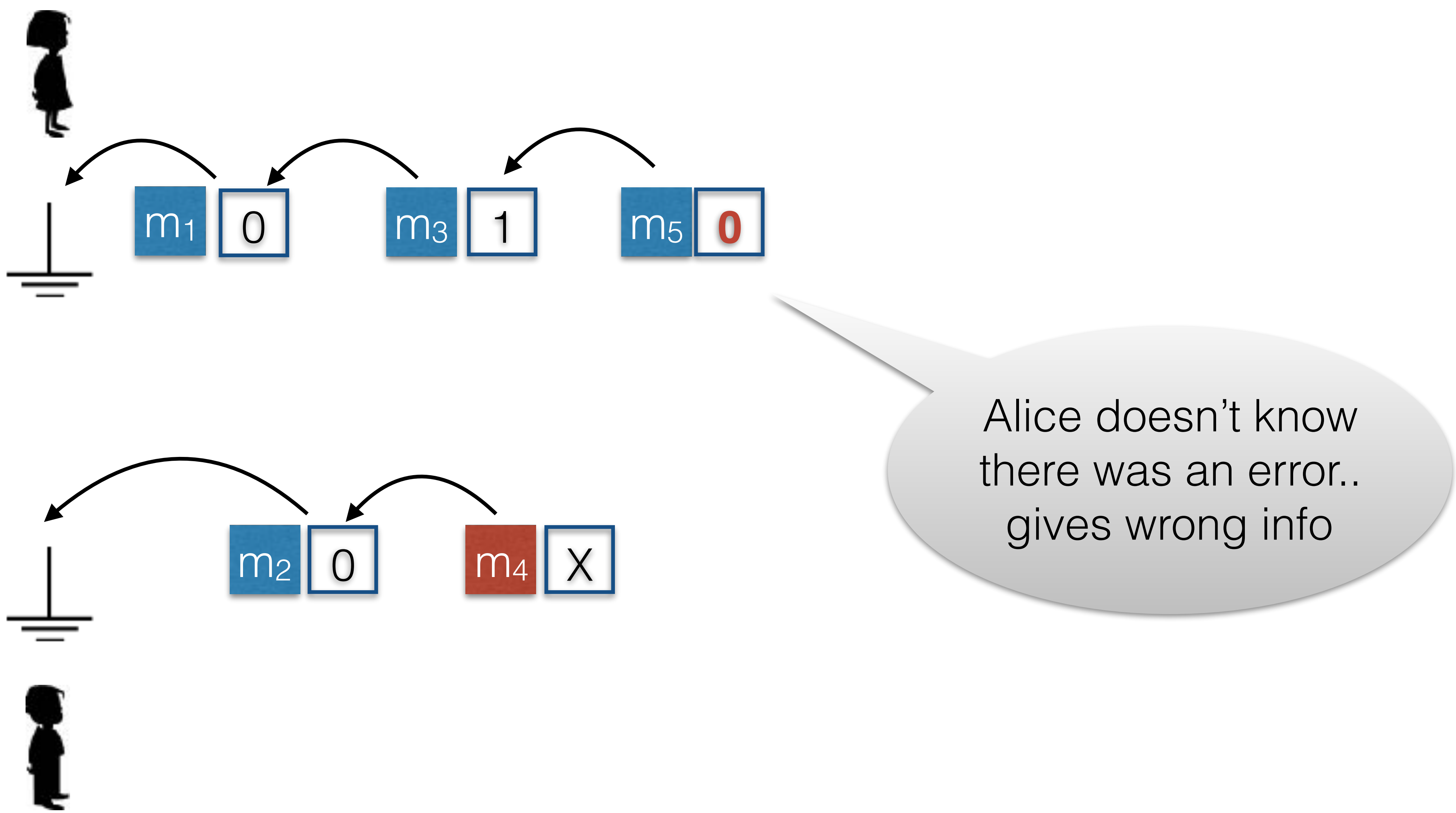
- We define a noisy ***KW mapping*** between formulas and protocols
  - Short Circuit noise == Channel noise (assuming feedback)
- The sender learns the received symbol via a “noiseless feedback “ channel



Coding Scheme - Overview

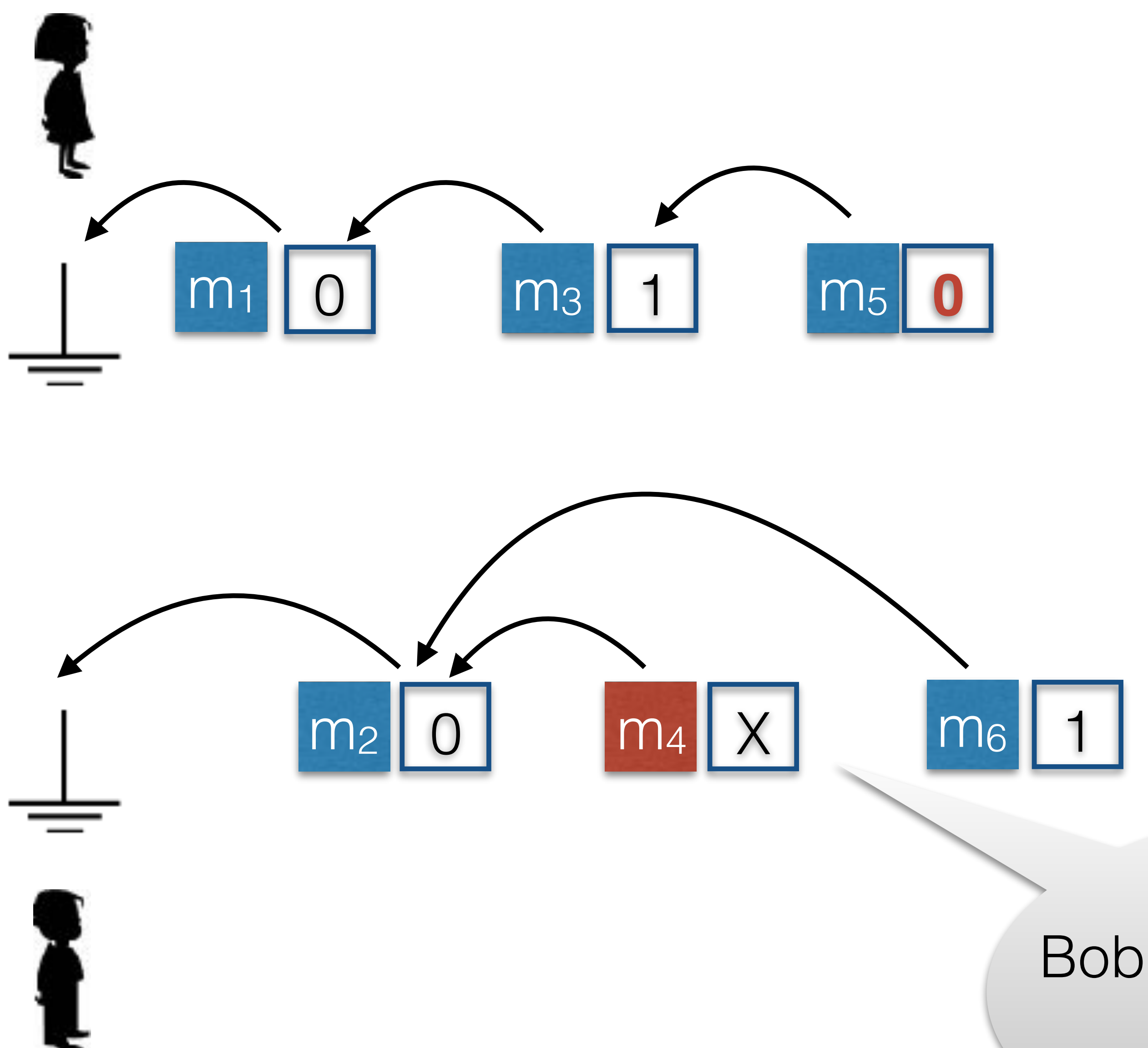
- Assume a noiseless binary protocol π
- Alice and Bob simulate π message by message.
  Each message contains:
  - the “next” bit according to π
  - a link to the previous non-corrupt message sent by the party (as learnt by feedback)
- Each received message induces a “chain” of *allegedly* correct messages. The next step follows this chain
- At the end, the longest chain is to be trusted

Coding Scheme



Aim: simulate the noiseless protocol step-by-step

Coding Scheme

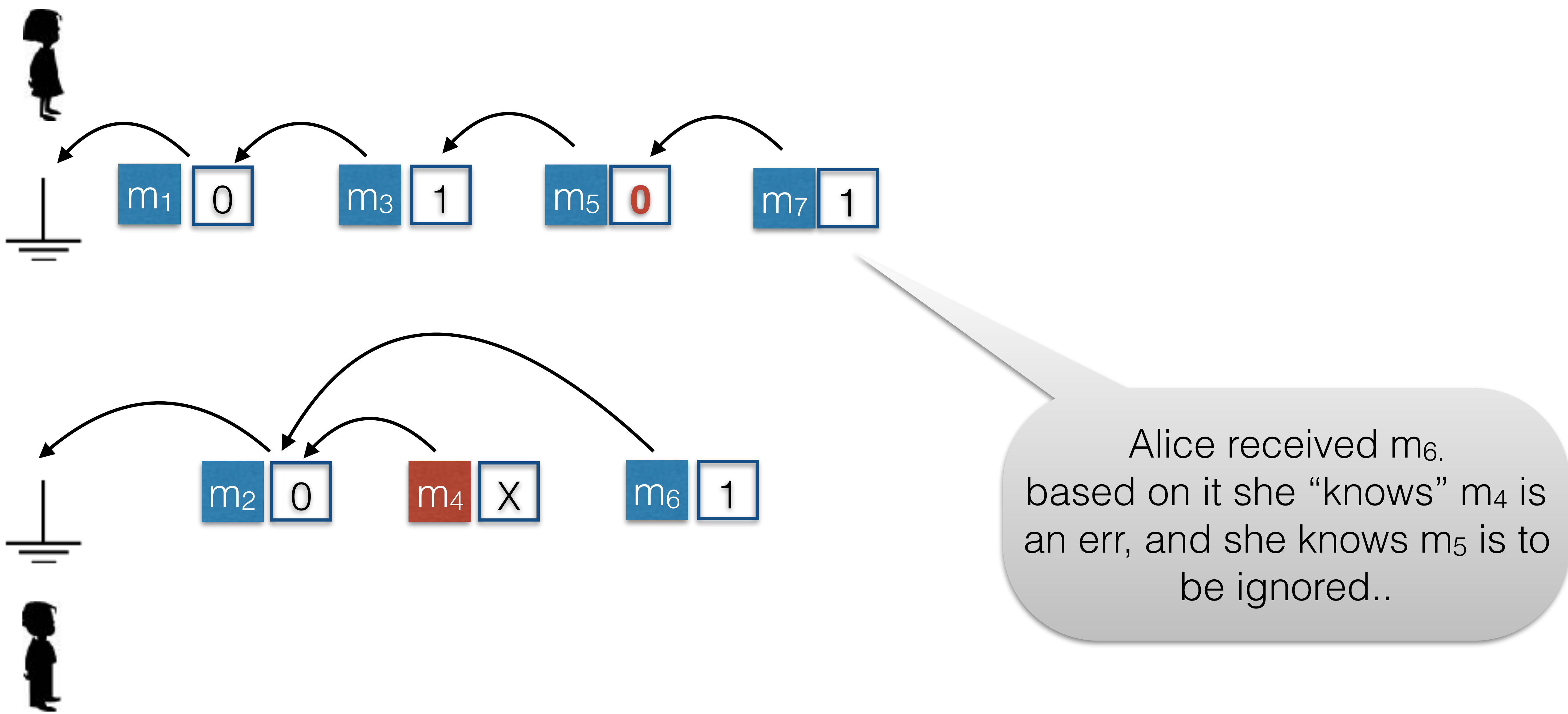


This extension **ignores** m_5,

Bob “knows” m_5 is based on err

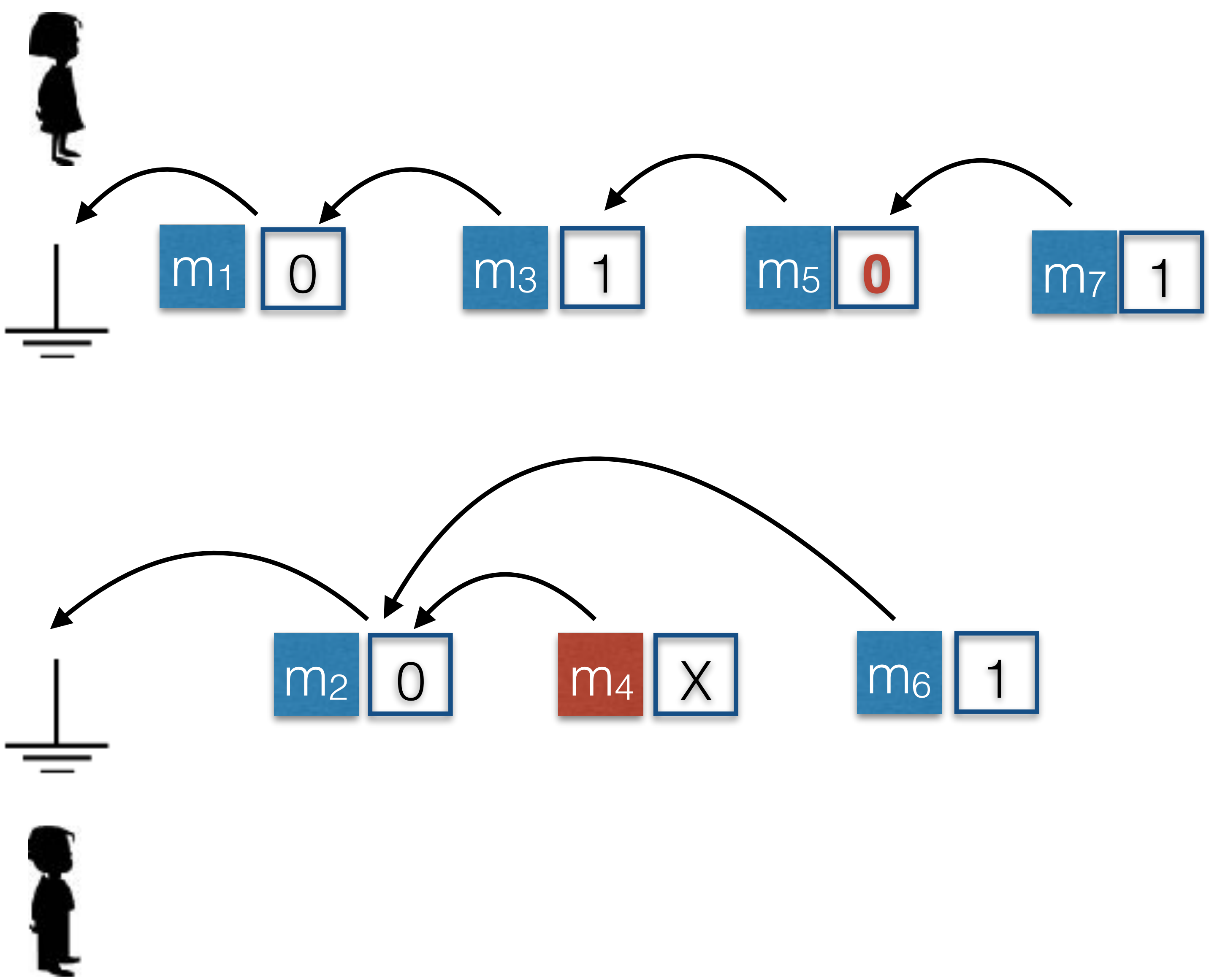
Bob knows this is wrong (via feedback)

Coding Scheme



Output: the transcript implied by the longest chain

Coding Scheme

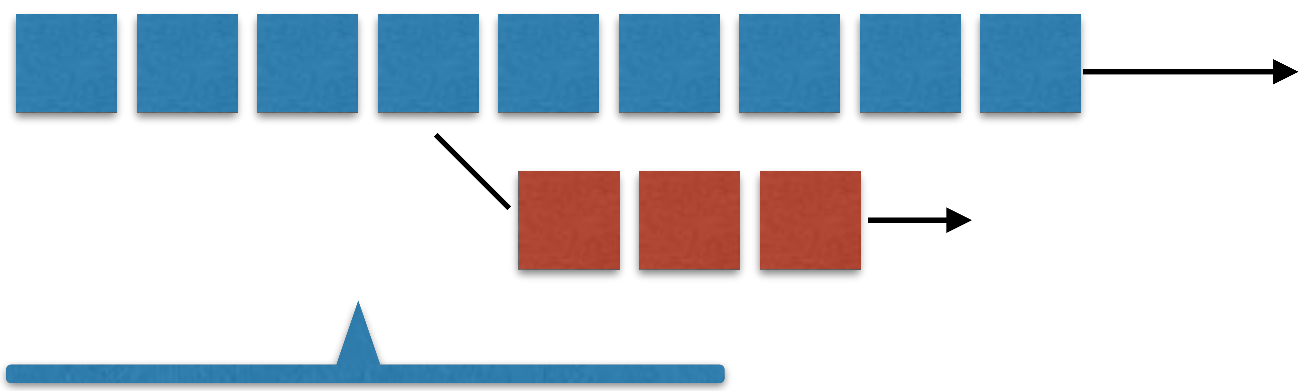


- Messages are **not** alternating order
  - the more noise on Bob's messages the **less** he gets to speak in the future

Attacks (1)

- Adversary may try to build its own chain
- But with 1/5-fraction corruptions, his chain will be shorter

Attacks (2)



(all the needed info is here)

- Adversary may incorrectly extend a correct chain
- But in order to make its chain the longest, it must start late
- by then, the chain's already simulated the entire transcript.

Techniques:
Lower Bound

Lower Bound

- Note, (1/5,1/5)-corruptions cannot fool protocols with exponential (blowup in) communication:
    - Use Shannon code with relative distance ≈ 1 to exchange the parties inputs.
    - Withstands noise rate of $\approx 1/2$ per direction of the channel

Lower Bound

- Yet, when the blow-up is restricted
(e.g., communication < size of the input) :

- By a Pigeon hole principle, we can show a function f and inputs x,y,x',y' for which

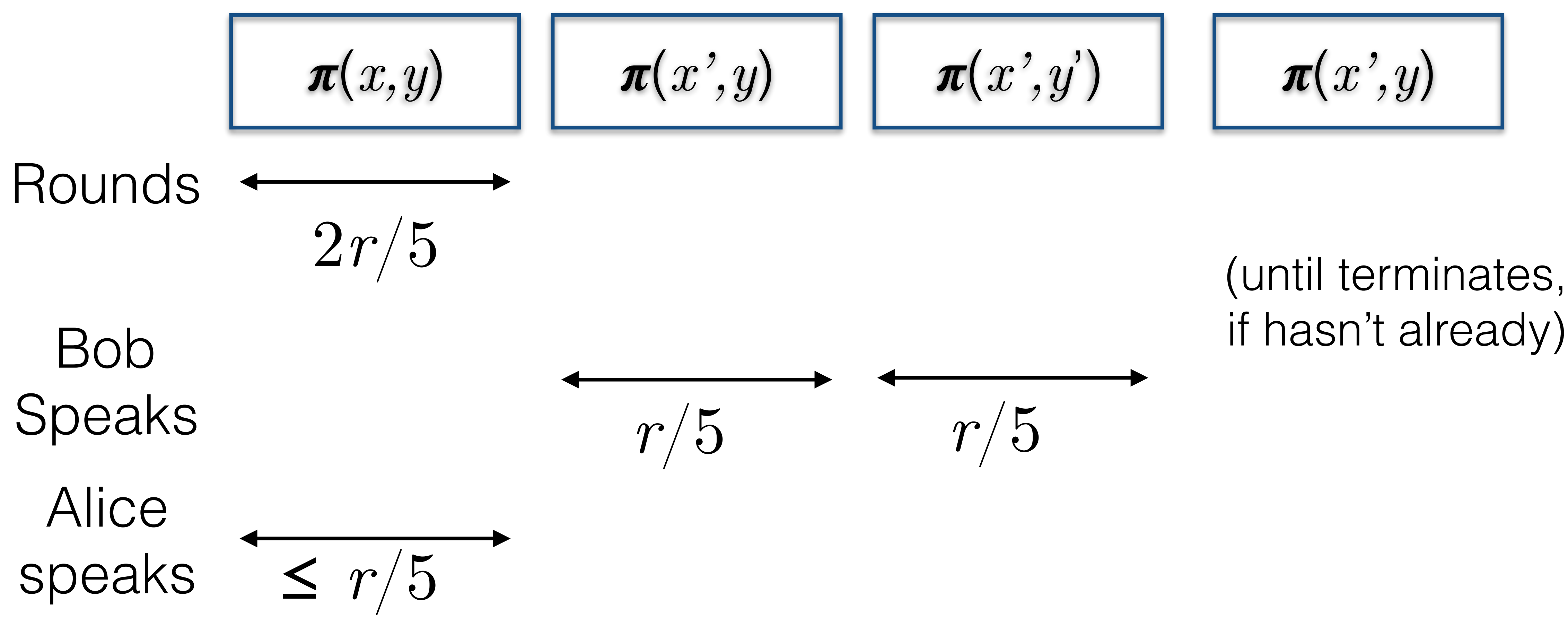
1. $f(x,y)$ ≠ $f(x',y)$ ≠ $f(x',y')$

- If the computation of f takes r rounds by some protocol, then during its first $2r/5$ rounds:

2. Alice (wlog) speaks at most <u>half</u> of the times
3. If Alice has x, then the protocol sends exactly the same messages whether Bob holds y or y'

Lower Bound

- Create the following confusing transcript:



- is a (1/5,1/5)-corruption of $\boldsymbol{\pi}(x',y)$ and one of $\{\boldsymbol{\pi}(x',y'),\ \boldsymbol{\pi}(x,y)\}$

Lower Bound

- Example: Assume $\boldsymbol{\pi}$ terminates before 4-th part

	Alice ➜ x		Bob ➜ y'
(x',y)	$\boldsymbol{\pi}(x,y)$	$\boldsymbol{\pi}(x',y)$	$\boldsymbol{\pi}(x',y')$
	Alice ➜ x	Bob ➜ y	
(x',y')	$\boldsymbol{\pi}(x,y)$	$\boldsymbol{\pi}(x',y)$	$\boldsymbol{\pi}(x',y')$

$\boldsymbol{\pi}(x,y)=\boldsymbol{\pi}(x,y')$

from (3) of pigeon hole

Since $f(x',y) \neq f(x',y')$

we are done

(1) of pigeon hole…

Lower Bound

- Problem:
  - Need to apply the above on *KW-relation*,
    rather than on a *function*.
  - $f(x', y) \neq f(x', y')$ translates to
    confusing Alice between $KW(x', y)$ and $KW(x', y')$
  - but maybe **both** are a correct output of the protocol?!
- We use KW relation of the *parity function* $par(x_1, \ldots, x_n) = x_1 \oplus \cdots \oplus x_n$,
  chosing inputs so that

$$KW_{par}(x', y) \cap KW_{par}(x', y') = \varnothing$$

Summary

Summary

- A two-directional “noisy” KW mapping between protocols and formulas

- Coding scheme with resilience $1/5$-ε (const alphabet)

  ➡ Formula resilient to ($1/5$-ε, $1/5$-ε)–noise

- Impossibility of coding with $1/5$ (*const rate*)

  ➡ No *small* formula is resilient to ($1/5$,$1/5$)-noise

Open Problems

1. The binary / fan-in2 case?

2. General faults: stuck to 0/1, flip, short-circuit

3. KW connects *formulas* with *2-party* protocols

    - Can we map general *circuits* with some kind of communication model?

    - (Branching Programs? multiparty protocols?)

The End…