Noise Stability is Low-Dimensional

Anindya De, Elchanan Mossel, Joe Neeman

Gaussian noise stability

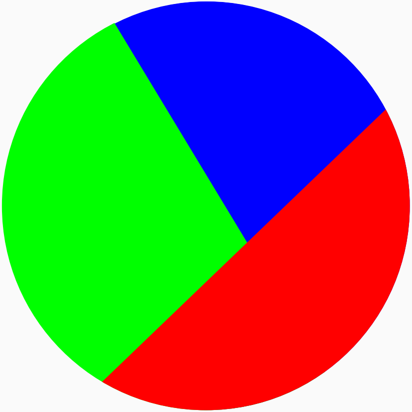
Take X and Y a pair of ρ-correlated Gaussians in $\mathbb{R}^n$ ($0 < \rho < 1$).

For a partition $A = (A_1, \ldots, A_k)$ of $\mathbb{R}^n$ into k parts, define

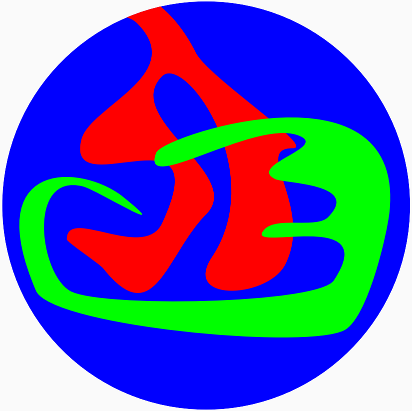
$$\mathrm{Stab}_\rho(A) = \Pr(X \text{ and } Y \text{ land in the same part})$$

Gaussian noise stability

Noise stable



Not noise stable

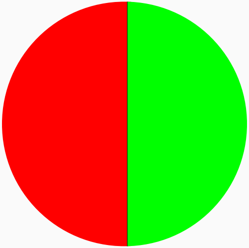


Theorem (Borell '85)

For a partition of $\mathbb{R}^n$ into two parts of equal Gaussian measure,

$$\mathrm{Stab}_\rho(A) \leq \frac{1}{2} + \frac{\sin^{-1}\rho}{\pi}.$$

Equality is attained for a partition into half-spaces.



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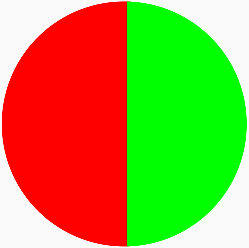

- well-known links to computational complexity (KKMO '05)

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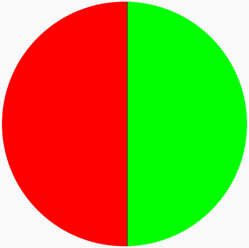

- well-known links to computational complexity (KKMO '05)
- one-dimensional phenomenon

Gaussian noise stability

Theorem (???)

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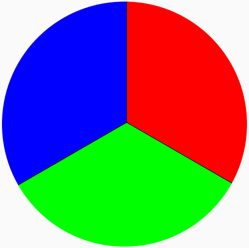
???

Gaussian noise stability

Conjecture (Peace sign conjecture)

The optimal partition looks like a peace sign.



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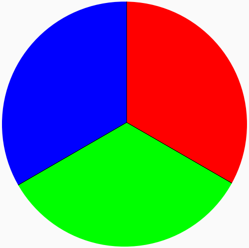

Gaussian noise stability

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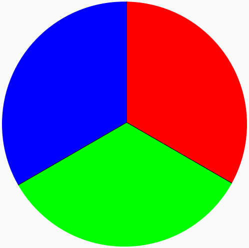

Conjecture (Multi-dimensional peace sign conjecture)

For partitions into k equal measures, the optimal partition occurs in $\mathbb{R}^{k-1}$. It looks like a multi-dimensional peace sign.

Gaussian noise stability

Theorem (De-Mossel-N.)
For any k and any $\epsilon > 0$, there is a computable $n_0 = n_0(k, \epsilon)$ such that an ϵ-approximately optimal partition occurs in $\mathbb{R}^{n_0}$.

Gaussian noise stability

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Corollary

The optimal value of k-part noise stability is computable.

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Corollary (sort of)

The non-interactive correlation distillation value with k-ary outputs is computable.

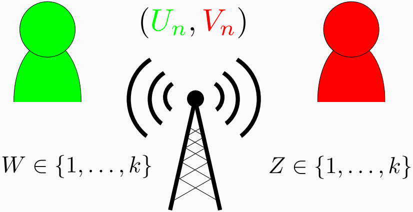
Correlation distillation

$$(U_1, V_1)$$

$$(U_2, V_2)$$

$$\vdots$$

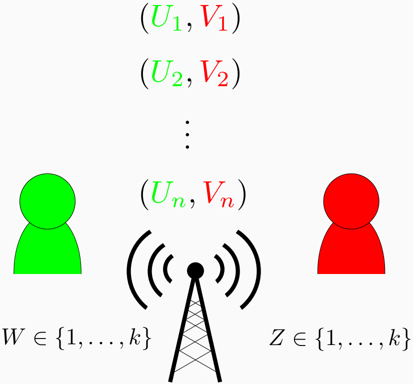
$$(U_n, V_n)$$



Goal: produce uniform output, agree with maximal probability.

What is the probability of agreement?

Correlation distillation



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Ghazi-Kamath-Sudan '16: reduction to correlated Gaussian signals

The main theorem

Theorem (De-Mossel-N.)

For any k and any $\epsilon > 0$, there is a computable $n_0 = n_0(k, \epsilon)$ such that an ϵ-approximately optimal partition occurs in $\mathbb{R}^{n_0}$.

Proof outline

Idea: take an optimal partition in $\mathbb{R}^n$ (n huge) and try to "simulate" it in $\mathbb{R}^{n_0}$.

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1. An optimal partition is close to a bounded-degree polynomial threshold function (PTF)
2. A bounded-degree PTF can be approximately simulated by a bounded-degree PTF on a bounded number of variables

Step 1: approximation by polynomials

Approximation by polynomials

Think of a partition as a function $f : \mathbb{R}^n \to \{e_1, \ldots, e_k\} \subset \mathbb{R}^k$.

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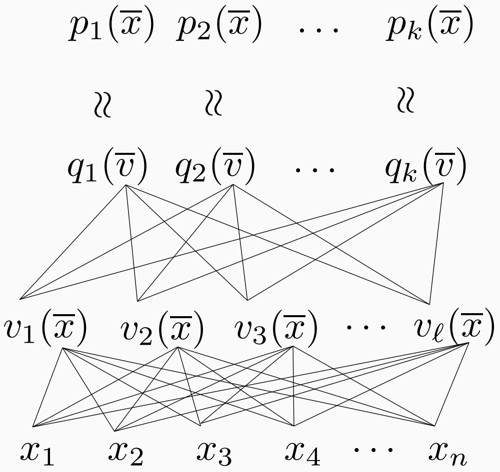
Real proof goes through a smoothing/rounding procedure, and a connection with Gaussian surface area (KNOW '15).

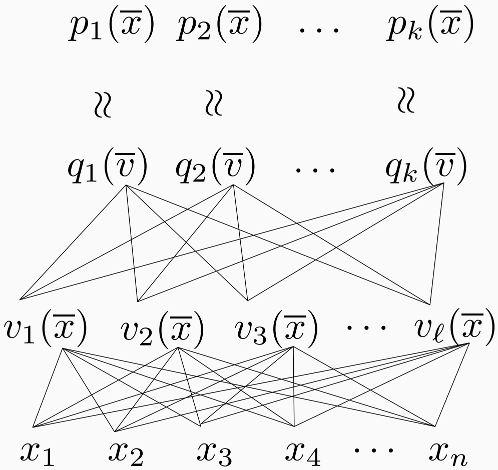
Step 2: dimension reduction

Polynomial structure theorem (De-Servedio)



where ℓ is bounded and $v_1, \dots, v_\ell$ are "nice"

Polynomial Central Limit Theorem (De-Servedio)



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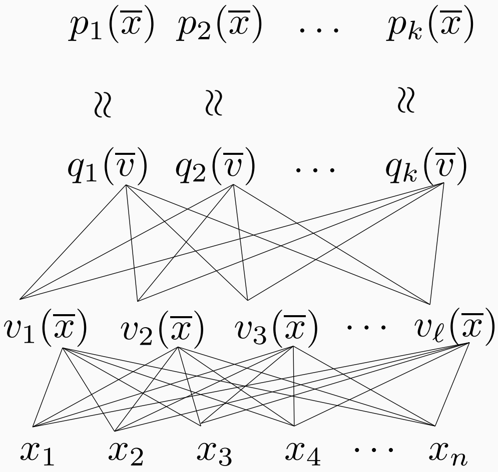

"Nice" polynomials satisfy a CLT, so they may as well be linear functions of ℓ variables

Polynomial Central Limit Theorem (De-Servedio)

$$p_1(\overline{x}) \quad p_2(\overline{x}) \quad \dots \quad p_k(\overline{x})$$

$$\wr\wr \qquad \wr\wr \qquad\qquad \wr\wr$$

$$q_1(\overline{v}) \quad q_2(\overline{v}) \quad \dots \quad q_k(\overline{v})$$

$$\sum_{i=1}^{\ell} a_{1,i}x_i \qquad \sum_{i=1}^{\ell} a_{2,i}x_i \qquad \sum_{i=1}^{\ell} a_{3,i}x_i \qquad \sum_{i=1}^{\ell} a_{\ell,i}x_i$$

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Open problem

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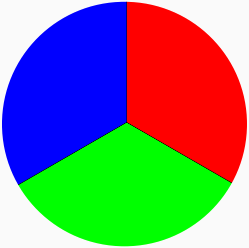

Thank you!