Resolution and the binary encoding of combinatorial principles

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Proof Complexity in Resolution over s-DNF

A 2-DNF: $((v_1 \wedge \neg v_2) \vee (v_2 \wedge v_3) \vee (\neg v_1 \wedge v_3))$

	Resolution (= Res(1))	Res(2)
Main Rule	$\dfrac{C \vee x \qquad \neg x \vee D}{C \vee D}$	$\dfrac{C \vee (x \wedge y) \qquad (\neg x \vee \neg y) \vee D}{C \vee D}$
Refutations for	CNF	CNF

Proof Size for UNSAT CNF: minimal number of s-DNFs to derive the empty clause $\Box$.

Proof Complexity in Resolution over s-DNF

1. The $\wedge$-*introduction rule* is

$$\frac{\mathcal{D}_1 \vee \bigwedge_{j \in J_1} l_j \quad \mathcal{D}_2 \vee \bigwedge_{j \in J_2} l_j}{\mathcal{D}_1 \vee \mathcal{D}_2 \vee \bigwedge_{j \in J_1 \cup J_2} l_j},$$

   provided that $|J_1 \cup J_2| \leq s$.

2. The *cut (or resolution) rule* is

$$\frac{\mathcal{D}_1 \vee \bigvee_{j \in J} l_j \quad \mathcal{D}_2 \vee \bigwedge_{j \in J} \neg l_j}{\mathcal{D}_1 \vee \mathcal{D}_2},$$

3. The two *weakening rules* are

$$\frac{\mathcal{D}}{\mathcal{D} \vee \bigwedge_{j \in J} l_j} \quad \text{and} \quad \frac{\mathcal{D} \vee \bigwedge_{j \in J_1 \cup J_2} l_j}{\mathcal{D} \vee \bigwedge_{j \in J_1} l_j},$$

   provided that $|J| \leq s$.

We turn a Res(s) proof upside-down, i.e. reverse the edges of the underlying graph and negate the s-DNF on the vertices, we get a special kind of restricted branching s-program whose nodes are labelled by s-CNFs and at each node some s-disjunction is queried.

1. Querying a new s-disjunction, and branching on the answer, which can be depicted as follows.

$$
\begin{array}{ccccc}
 & & \mathcal{C} & & \\
 & & ?\bigvee_{j\in J} l_j & & \\
 & \top \swarrow & & \searrow \bot & \\
\mathcal{C} \wedge \bigvee_{j\in J} l_j & & & & \mathcal{C} \wedge \bigwedge_{j\in J} \neg l_j
\end{array}
\tag{1}
$$

2. Querying a known s-disjunction, and splitting it according to the answer:

$$
\begin{array}{ccccc}
 & & \mathcal{C} \wedge \bigvee_{j\in J_1\cup J_2} l_j & & \\
 & & ?\bigvee_{j\in J_1} l_j & & \\
 & \top \swarrow & & \searrow \bot & \\
\mathcal{C} \wedge \bigvee_{j\in J_1} l_j & & & & \mathcal{C} \wedge \bigvee_{j\in J_2} l_j
\end{array}
\tag{2}
$$

3. There are two ways of forgetting information,

$$\begin{array}{c} \mathcal{C}_1 \wedge \mathcal{C}_2 \\ \downarrow \\ \mathcal{C}_1 \end{array} \qquad \text{and} \qquad \begin{array}{c} \mathcal{C} \wedge \bigvee_{j \in J_1} l_j \\ \downarrow \\ \mathcal{C} \wedge \bigvee_{j \in J_1 \cup J_2} l_j \end{array} , \tag{3}$$

k-clique principle

$G = (V, E)$. We want to define a formula

$\mathrm{Clique_k}(G)$ satisfiable iff G contains a k-clique.

$x_{iv} \equiv$ "v is the i-th node in the clique"

$$\mathrm{Clique_k}(G) = \begin{cases} \bigvee_{v \in V} x_{i,v} & i \in [k] & \text{a node in each position} \\ \neg x_{i,v} \vee \neg x_{i,u} & u \neq v \in V, i \in [k] & \text{no two nodes in one position} \\ \neg x_{i,u} \vee \neg x_{j,v} & (u,v) \notin E, i \neq j \in [k] & \text{"no-edges" are not in the clique} \end{cases}$$

Fact

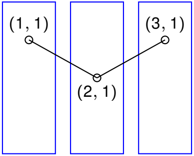
$\mathrm{Clique_k}(G)$ *UNSAT iff G does not have a k-clique* *k*-Clique Principle: Simplified version

- G formed from k blocks V_b of n nodes each: $G = (\bigcup_{b\in[k]} V_b, E)$
- Variables $v_{i,q}$ with $i \in [k], a \in [n]$, with clauses

$$\text{Clique}_k^n(G) = \begin{cases} \neg v_{i,a} \vee \neg v_{j,b} & ((i,a),(j,b)) \notin E \\ \bigvee_{a\in[n]} v_{i,a} & i \in [k] \end{cases}$$

Fact

$\text{Clique}_k^n(G)$ *UNSAT iff G does not have a k-clique*



$$\mathrm{Clique}^{n}_{k}(G) = \begin{cases} x_{1,1} \\ x_{2,1} \\ x_{3,1} \\ (\neg x_{1,1} \vee \neg x_{3,1}) \end{cases}$$

Motivations(Informal): Clique^{n}_{k} captures the proof strength of adding to a proof system the ability to count up to k. [1,2]

[1]=[Beyersorff Galesi Lauria Razborov 12]
[2]=[Dantchev Martin Szeider 11]

k-Clique Principle (Binary Version):

- (Bit-)Variables: $\omega_{i,j}$, for $i \in [k]$, $j \in [\log n]$
- Notation:

$$\omega_{i,j}^{a_j} = \begin{cases} \omega_{i,j} & \text{if } a_j = 1 \\ \neg\omega_{i,j} & \text{if } a_j = 0 \end{cases}$$

$$v_{i,j} \equiv (\omega_{i,1}^{a_1} \wedge \ldots \wedge \omega_{i,\log n}^{a_{\log n}}), \text{ where } (j)_2 = \vec{a}$$

$$\text{Bin-Clique}_{\text{k}}^{\text{n}}(G) = \bigwedge_{((i,a),(j,b)) \notin E} \left((\omega_{i,1}^{1-a_1} \vee \ldots \vee \omega_{i,\log n}^{1-a_{\log n}}) \vee (\omega_{j,1}^{1-b_1} \vee \ldots \vee \omega_{j,\log n}^{1-b_{\log n}}) \right)$$

- preserve the combinatorial hardness of the unary principle;
- are less exposed to details of the encoding when attacked with a lower bound technique;
- give significative lower bounds.

PHP^m_n : *Unary encoding*

$\bigvee_{j=1}^{n} v_{i,j} \quad i \in [m]$

$\overline{v}_{i,j} \vee \overline{v}_{i',j} \quad i, \neq i' \in [m], j \in [n]$

Bin-PHP^m_n : *Binary encoding*

$\bigvee_{j=1}^{\log n} \neg\omega_{i,j} \vee \bigvee_{j=1}^{\log n} \neg\omega_{i',j} \quad i \neq i' \in [m]$

Size-Width tradeoffs for Res: $Size(F \vdash) \geq e^{\Omega\left(\frac{(w(F\vdash) - w(F))^2}{Vars(F)}\right)}$

Space-Width Relations for Res: $Space(F \vdash) \geq w(F \vdash) - w(F) + 1$

$w(\mathrm{PHP}) = n$ while $w(\text{Bin-PHP}) = 2logn$

Fact

Res(1) *proofs of* $\mathrm{Clique}^{\mathrm{n}}_{\mathrm{k}}(G) \longmapsto$ Res($\log n$) *proofs of* $\text{Bin-Clique}^{\mathrm{n}}_{\mathrm{k}}(G)$.

$$v_{i,a} \equiv (\omega_{i,1}^{a_1} \wedge \ldots \wedge \omega_{i,\log n}^{a_{\log n}})$$

Fact

Res(1) *proofs of* $\mathrm{PHP}^m_n \longmapsto$ Res($\log n$) *proofs of* Bin-PHP^m_n

Known results for k-Clique Principles in Res

- For any G there are $O(n^k)$ proofs in tree-Res (brute force)
- If G is the $(k-1)$-partite graph: $\mathrm{Clique}^n_k(G)$ has Reg-Res refutations of size $O(2^k n^2)$ [1]
- Difficult to find G's without a k-clique making hard to refute $\mathrm{Clique}^n_k(G)$.

Known Lower Bounds: $(G \sim \mathcal{G}(n,p),\ p = n^{-\frac{2(1+\epsilon)}{k-1}})$

$G \sim \mathcal{G}(n,p)$	tree-Res	Reg-Res	Res(1)	Res(s)
$\mathrm{Clique}^n_k(G)$	$\Omega(n^k)$[1]	$\Omega(n^k)$[2]	Open - $\Omega(2^k)$ [4]	Open
$\mathrm{Bin\text{-}Clique}^n_k(G)$	–	–	$\Omega(n^k)$[3]	$\Omega(n^k)$, $s = o(\log\log n)$

[1] = [Beyersdorff Galesi Lauria 13]
[2] = [Atserias Bonacina de Rezende Lauria Nördstrom Razborov 18]
[3] = [Lauria Pudlák Rödl Thapen 17]
[4] = [Pang 19, ECCC]

Results for Bin-PHP$_n^m$

Theorem

$\epsilon, \delta > 0$. *Any refutation of* Bin-PHP$_n^m$ *in* Res(s) *for* $s \leq \sqrt[2+\epsilon]{\log n}$ *is of size* $2^{\Omega(n^{1-\delta})}$.

Theorem

*There are tree-*Res(1) *refutations of* Bin-PHP$_n^m$ *of size* $2^{\Theta(n)}$.

Lower Bound Proof (for Bin-Clique$^n_k(G)$)

Main Tools(for Binary Principles):

1. *Covering Number* on s-DNFs [1]
   - Res(s) proofs with small CN efficiently simulated in Res($s-1$)
   - *Bottlenecks*
2. *(Random) restrictions* for binary principles
3. *Hardness properties* of Bin-Clique$^n_k(G)$, when $G \sim \mathcal{G}(n,p)$ [2,3,4]
4. Induction on s.
   - Base Case: known hardness on Res(1) [4].

[1]=[Segerlind Buss Impagliazzo 04]
[2]=[Beyersdorff Galesi Lauria 13]
[3]=[Atserias Bonacina de Rezende Lauria Nördstrom Razborov 18]
[4]=[Lauria Pudlák Rödl Thapen 17]

Covering number of a Res(s) proof

A *covering set* for a s-DNF $\mathcal{F}$ is a set of literals L such that each term of $\mathcal{F}$ has at least a literal in L.

The *covering number* $cv(\mathcal{F})$ of a s-DNF $\mathcal{F}$ is the minimal size of a covering set for $\mathcal{D}$.

$$CN(\pi) = \max_{\mathcal{F} \in \pi} c(\mathcal{F})$$

Lemma (Simulation Lemma)

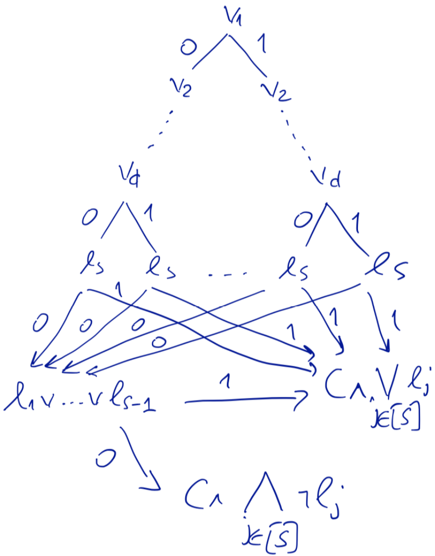
If F has a size N refutation π in $\mathrm{Res}(s)$ with $CN(\pi) < d$, then F has a $\mathrm{Res}(s-1)$ refutation of size at most $2^{d+2}N$.

Put π upside-down. Get a restricted branching s-program whose nodes are labelled by s-CNFs and at each node some s-disjunction $\bigvee_{j\in[s]} l_j$ is queried.

Example

$$\begin{array}{ccc} & \vdots & \\ & \mathcal{C} & \\ & ?\bigvee_{j\in[s]} l_j & \\ 1 \swarrow & & \searrow 0 \\ \mathcal{C}\wedge\bigvee_{j\in[s]} l_j & & \mathcal{C}\wedge\bigwedge_{j\in[s]} \neg l_j \end{array} \tag{4}$$

Let $cv(\mathcal{C}) < d$, witnessed by variable set $\{v_1, \ldots, v_d\}$.



Bottlenecks in Res(s)

A *c*-bottleneck in a Res(s) proof is a s-DNF F whose $cv(F) \geq c$. $c(s)$ is the *bottleneck number* at Res(s).

Fact (Independence)

If $c = rs$, $r \geq 1$ and $cv(F) \geq c$, then in F it is always possible to find r pairwise disjoint s-tuples of literals $T_1 = (\ell_1^1, \ldots, \ell_1^s), \ldots, T_r = (\ell_r^1, \ldots, \ell_r^s)$ such that the $\bigwedge T_i$'s are terms of F.

Restrictions

A *s-restriction* assigns $\lfloor \frac{\log n}{2^{s+1}} \rfloor$ bit-variables $\omega_{i,j}$ in each block $i \in [k]$.

Fact

if σ and τ are (disjoint) s-restrictions, then $\sigma\tau$ is a $(s-1)$-restriction

A *random s-restriction* for $\text{Bin-Clique}^{n}_{k}(G)$ is an s-restriction obtained by choosing independently in each block i, $\lfloor \frac{\log n}{2^{s+1}} \rfloor$ variables among $\omega_{i,1}, \ldots, \omega_{i,\log n}$, and setting these uniformly at random to 0 or 1.

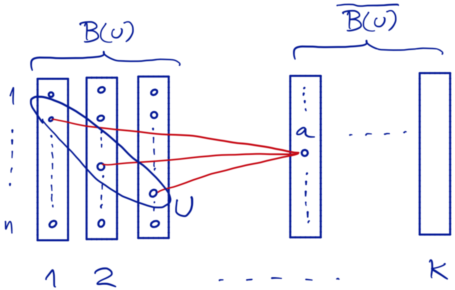
Hardness Properties

$G = (\bigcup_{b \in [k]} V_b, E)$ and $0 < \alpha < 1$. U is *α-transversal* if:

1. $|U| \leq \alpha k$, and
2. for all $b \in [k]$, $|V_b \cap U| \leq 1$.

Let $B(U) \subseteq [k]$ be the set of blocks mentioned in U, and $\overline{B(U)} = [k] \setminus B(U)$.

U is *extendible* in a block $b \in \overline{B(U)}$ if there exists a vertex $a \in V_b$ which is a *common neighbour of all nodes in U*.



A restriction σ is *consistent* with $v = (i, a)$ if for all $j \in [\log n]$, $\sigma(\omega_{i,j})$ is either a_j or not assigned (i.e. assigns the right bit or can do it in the future)

Definition

Let $0 < \alpha, \beta < 1$. A α-transversal U is β-extendible, if for all β-restriction σ, there is a node v^b in each block $b \in \overline{B(U)}$, such that σ is consistent with v^b.

Lemma (Extension Lemma, similar to [1])

Let $0 < \epsilon < 1$*, let* $k \leq \log n$*. Let* $1 > \alpha > 0$ *and* $1 > \beta > 0$ *such that* $1 - \beta > \alpha(2 + \epsilon)$*. Let* $G \sim \mathcal{G}^(n, p)$*. With high probability both properties hold:*

1. *all* α*-transversal sets* U *are* β*-extendible;*
2. *G does not have a k-clique.*

[1]=[Beyersodrff Galesi Lauria 13]

Idea of the proof

Property (Clique(G, s, k))

For any s-restriction ρ, there are no Res(s) refutations of $\text{Bin-Clique}^n_k(G)\restriction_\rho$ *of size less than* $n^{\frac{\delta(k-1)}{d(s)}}$.

Theorem

If $\text{Clique}(G, s, k)$ *holds, then there are no* $\text{Res}(s)$ *proofs of* $\text{Bin-Clique}^n_k(G)$ *with size* $n^{\frac{\delta(k-1)}{d(s)}}$.

By Extension Lemma there exists a $G \sim \mathcal{G}(n, n^{-\frac{2(1+\epsilon)}{(k-1)}})$ with the extension properties.

Lemma

$\text{Clique}(G, 1, k)$ *holds. (use [1])*

[1]=[Lauria Pudlák Rödl Thapen 17]

Steps of the proof

Lemma

Clique$(G, s-1, k) \Rightarrow$ Clique(G, s, k).

We prove that $\neg$Clique$(G, s, k) \Rightarrow \neg$Clique$(G, s-1, k)$. Let $L(s) = n^{\frac{\delta(k-1)}{\mathrm{d}(s)}}$.

- Since $\neg$Clique(G, s, k), then $\exists$ a s-restriction ρ and π a proof of Bin-Clique$^{\mathrm{n}}_{\mathrm{k}}(G)\restriction_\rho$, such that $|\pi| < L(s)$.
- Let $c = c(s)$ be the bottleneck number and $r = cs$
- σ be a s-random restriction on Bin-Clique$^{\mathrm{n}}_{\mathrm{k}}(G)\restriction_\rho$.
- Pr[bottleneck F survives in $\pi\restriction_\sigma] \leq e^{-\frac{r}{\mathrm{p}(s)}}$. Use *Independence Property*.
- $\Pr[CN(\pi\restriction_\sigma) \geq c] < 1$. Union bound.
- Define $\tau = \sigma\rho$ and apply *Simulation Lemma* to $\pi\restriction_\sigma$. We get a (s-1)-restriction τ and a $\leq L(s)2^{c+2}$ size proof in *Res*$(s-1)$ of Bin-Clique$^{\mathrm{n}}_{\mathrm{k}}(G)\restriction_\tau$. If $L(s)2^{c+2} < L(s-1)$, this is $\neg$Clique$(G, s-1, k)$.
- knowing $\mathrm{p}(s)$, define $\mathrm{d}(s)$ and $c(s)$ in such a way to force $L(s)2^{c+2} < L(s-1)$ and union bound to work.

The case of Bin-PHP$_n^m$

	tree-Res	Res(s), $m \leq 2n$	Res(s), $m > 2n$
Bin-PHP$_n^m$	$2^{\Theta(n)}$	$2^{\Omega(n^{1-\delta})}$ $(s \leq \log^{\frac{1}{2+\epsilon}} n)$	
PHP$_n^m$	$2^{\Theta(n \log n)}$ [3,4]	$2^{\Omega(\frac{n}{\log \log n})} (s \leq \sqrt{\log n})$ [2]	[1]

A form of optimality of the lower bound: [5] Proved an upper bound of $O(2^{\sqrt{n \log n}})$ in Res for PHP$_n^m$, when $m \geq 2^{\sqrt{n \log n}}$. Use the fact that size S proof in Res(1) for PHP implies size S proof in Res($\log n$) for Bin-PHP.

[1]=[Razborov 02] (Survey: "Proof Complexity of PHP")
[2]=[Segerlind Buss Impagliazzo 03]
[3]=[Beyersdorff Galesi Lauria 10]
[4]=[Dantchev Riis 01]
[5]=[Buss Pitassi 97]

Other Results for Binary Principles

$\mathrm{OP_n}$: *Unary encoding*

$\overline{v}_{x,x}$	$x \in [n]$
$\overline{v}_{x,y} \vee \overline{v}_{y,z} \vee v_{x,z}$	$x, y, z \in [n]$
$\bigvee_{i\in[n]} v_{x,i}$	$x \in [n]$

$\mathrm{Bin\text{-}OP_n}$: *Binary encoding*

$\overline{\nu}_{x,x}$	$x \in [n]$
$\overline{\nu}_{x,y} \vee \overline{\nu}_{y,z} \vee \nu_{x,z}$	$x, y, z \in [n]$
$\bigvee_{i\in[\log n]} \omega_{x,i}^{1-a_i} \vee \nu_{x,a}$	$x, a \in [n]$

Lemma

$\mathrm{Bin\text{-}OP_n}$ *and* $\mathrm{Bin\text{-}LOP_n}$ *have polynomial size* $\mathrm{Res}(1)$ *proofs.*

- Res proof complexity of binary version of propositional version of principles which are expressible as first order formulae with no finite model in Π_2-form, i.e. as $\forall\vec{x}\exists\vec{w}\varphi(\vec{x},\vec{w})$ (Riis approach).
- Relations between different forms of binary encodings.
- Complexity of proofs in Res of a the binary versions of a large family of formulas (those having clauses $v_{i,j} \oplus v_{j,i}$, implying a comparisons among all pair of variables). *LOP* is included here.
- Comparisons of binary encodings with other compact encodings: unary functional encodings where i.e. clauses of the form $v_{i,1} \vee \ldots \vee v_{i,n}$ replaced with $v_{i,1} + \ldots + v_{i,n} = 1$.

Conclusions

We prove lower bounds for Res(s) for binary principles without using any form of the Switching Lemma.

Ad hoc random restrictions and an inductive argument allow to lift hardness results for Resolution.

- Binary versions of combinatorial principles might be useful benchmark/starting-point for trying lower bounds in stronger proof systems.

Thanks

Thanks for your attention!!!