From DNF compression to sunflower theorems via regularity

Jiapeng Zhang

UC San Diego $\to$ Harvard University

July 20, 2019

Joint works with Shachar Lovett and Noam Solomon

Overview

The Sunflower structure

Definition (Set systems)

Let X be a set. A set system $\mathcal{F} = \{S_1, \ldots, S_m\}$ on X is a collection of subsets of X. We call $\mathcal{F}$ a w-set system if each set of $\mathcal{F}$ has size at most w.

The Sunflower structure

Definition (Set systems)

Let X be a set. A set system $\mathcal{F} = \{S_1, \ldots, S_m\}$ on X is a collection of subsets of X. We call $\mathcal{F}$ a w-set system if each set of $\mathcal{F}$ has size at most w.

Definition (Sunflower, Erdős and Rado)

Given r sets $S_1, \ldots, S_r \subseteq X$ where $r \geq 3$. Denote as

$$B = S_1 \cap \cdots \cap S_r$$

We say it is a *r-sunflower* if for any $i, j \in [r]$, $S_i \cap S_j = B$.

The Sunflower structure

Definition (Set systems)

Let X be a set. A set system $\mathcal{F} = \{S_1, \ldots, S_m\}$ on X is a collection of subsets of X. We call $\mathcal{F}$ a w-set system if each set of $\mathcal{F}$ has size at most w.

Definition (Sunflower, Erdős and Rado)

Given r sets $S_1, \ldots, S_r \subseteq X$ where $r \geq 3$. Denote as

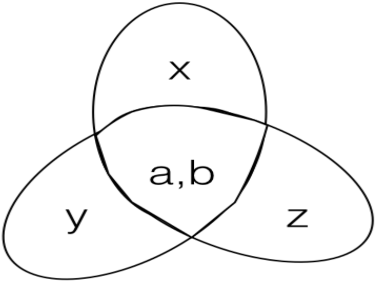
$$B = S_1 \cap \cdots \cap S_r$$

We say it is a *r-sunflower* if for any $i, j \in [r]$, $S_i \cap S_j = B$. We call B the kernal of this sunflower. In this talk, we focus on $r = 3$.

Sunflowers: an example

An example

Let $\mathcal{F} = \{\{a, b, x\}, \{a, b, y\}, \{a, b, z\}\}$. Then $\mathcal{F}$ is a 3-sunflower. Its kernal is $\{a, b\}$.



The sunflower lemma (conjecture)

Lemma (Erdős and Rado)

Let $\mathcal{F}$ be a w-set system such that $|\mathcal{F}| \geq w! \cdot 2^w$, then it contains a sunflower of size 3.

The sunflower lemma (conjecture)

Lemma (Erdős and Rado)

Let $\mathcal{F}$ be a w-set system such that $|\mathcal{F}| \geq w! \cdot 2^w$, then it contains a sunflower of size 3. Notice that $w! \approx w^w$.

The sunflower lemma (conjecture)

Lemma (Erdős and Rado)

Let $\mathcal{F}$ be a w-set system such that $|\mathcal{F}| \geq w! \cdot 2^w$, then it contains a sunflower of size 3. Notice that $w! \approx w^w$.

Conjecture (Erdős and Rado)

There is a constant $C > 0$ such that for any w-set system $\mathcal{F}$ with $|\mathcal{F}| \geq C^w$, it contains a sunflower of size 3.

DNF sparsification

Disjunctive Normal Forms

Definition

A DNF (Disjunctive Normal Form) is disjunction of conjunctive terms. The size of a DNF is the number of terms, and the width of a DNF is the maximal number of literals in a term.

Example

The function $f = (x_1 \wedge x_2) \vee (x_2 \wedge x_3 \wedge x_4) \vee (x_1 \wedge x_4) \vee (x_2 \wedge x_5)$ is a DNF of size 4 and width 3.

Set systems and monotone DNFs

A DNF is *monotone* if it contains no negated variables. Monotone DNFs are in one-to-one correspondence with set systems. Formally,

$$\mathcal{F} = \{S_1, \ldots, S_m\} \iff f_{\mathcal{F}}(x) := \bigvee_{j \in [m]} \bigwedge_{x_i \in S_j} x_i .$$

Set systems and monotone DNFs

A DNF is *monotone* if it contains no negated variables. Monotone DNFs are in one-to-one correspondence with set systems. Formally,

$$\mathcal{F} = \{S_1, \ldots, S_m\} \iff f_{\mathcal{F}}(x) := \bigvee_{j \in [m]} \bigwedge_{x_i \in S_j} x_i.$$

$$\{x_1.x_2\}, \{x_2, x_3, x_4\}, \{x_1, x_4\}, \{x_2, x_5\}$$
$$\iff (x_1 \wedge x_2) \vee (x_2 \wedge x_3 \wedge x_4) \vee (x_1 \wedge x_4) \vee (x_2 \wedge x_5)$$

DNF compression from (approximate) sunflowers

Theorem (Gopalan-Meka-Reingold)

Let f be a width-w DNF. Then for every $\varepsilon > 0$ there exist two width-w DNFs, f_{lower} and f_{upper} such that

(i) $f_{lower}(x) \leq f(x) \leq f_{upper}(x)$ *for all* x.

(ii) $\Pr[f_{lower}(x) \neq f_{upper}(x)] \leq \varepsilon$ *for a uniform random* x.

(iii) f_{lower} *and* f_{upper} *have size* $(w \log(1/\varepsilon))^{O(w)}$.

DNF compression from (approximate) sunflowers

Theorem (Gopalan-Meka-Reingold)

Let f be a width-w DNF. Then for every $\varepsilon > 0$ there exist two width-w DNFs, f_{lower} and f_{upper} such that

(i) $f_{lower}(x) \leq f(x) \leq f_{upper}(x)$ *for all* x.

(ii) $\Pr[f_{lower}(x) \neq f_{upper}(x)] \leq \varepsilon$ *for a uniform random* x.

(iii) f_{lower} *and* f_{upper} *have size* $(w \log(1/\varepsilon))^{O(w)}$.

Remark:

The term w^w comes Rossman's approximate sunflower bound.

DNF compression beyond sunflowers

Theorem (Lovett-Zhang)

Let f be a width-w DNF. Then for every $\varepsilon > 0$ there exists a width-w DNFs f_{lower} such that

(i) $f_{lower}(x) \leq f(x)$ *for all* x.

(ii) $\Pr[f_{lower}(x) \neq f(x)] \leq \varepsilon$ *for a uniform random* x.

(iii) f_{lower} *has size* $(1/\varepsilon)^{O(w)}$.

DNF compression beyond sunflowers

Theorem (Lovett-Zhang)

Let f be a width-w DNF. Then for every $\varepsilon > 0$ there exists a width-w DNFs f_{lower} such that

(i) $f_{lower}(x) \leq f(x)$ *for all* x.

(ii) $\Pr[f_{lower}(x) \neq f(x)] \leq \varepsilon$ *for a uniform random* x.

(iii) f_{lower} *has size* $(1/\varepsilon)^{O(w)}$.

Conjecture

Could we find a f_{upper} with the same bound as f_{lower}.

Regular set systems

Regular set systems

Definition

Let $\mathcal{F}$ be a set system on X, and $\kappa > 0$. We say that $\mathcal{F}$ is κ-regular if for every $A \subseteq X$,

$$\frac{|\{S \in \mathcal{F} : A \subseteq S\}|}{|\mathcal{F}|} \leq \kappa^{-|A|}$$

That is, each variable is in $1/\kappa$ fraction sets, each pair in $1/\kappa^2$ fraction sets, and so on.

Regular set systems

Definition

Let $\mathcal{F}$ be a set system on X, and $\kappa > 0$. We say that $\mathcal{F}$ is κ-regular if for every $A \subseteq X$,

$$\frac{|\{S \in \mathcal{F} : A \subseteq S\}|}{|\mathcal{F}|} \leq \kappa^{-|A|}$$

That is, each variable is in $1/\kappa$ fraction sets, each pair in $1/\kappa^2$ fraction sets, and so on.

We want to find a minimum κ so that any κ-regular set system contains a three sunflower.

Regular set systems

Definition

Let $\mathcal{F}$ be a set system on X, and $\kappa > 0$. We say that $\mathcal{F}$ is κ-regular if for every $A \subseteq X$,

$$\frac{|\{S \in \mathcal{F} : A \subseteq S\}|}{|\mathcal{F}|} \leq \kappa^{-|A|}$$

That is, each variable is in $1/\kappa$ fraction sets, each pair in $1/\kappa^2$ fraction sets, and so on.

We want to find a minimum κ so that any κ-regular set system contains a three sunflower.If $\mathcal{F}$ is not κ-regular, then we can apply induction on $\mathcal{F}_A := \{S \setminus A : (S \in \mathcal{F}) \wedge (A \subseteq S)\}$

From DNF sparsification to sunflowers

Conjecture: Upper bound DNF sparsification

Let f be a width-w monotone DNF. Then for every $\varepsilon > 0$ there exists a monotone width-w DNFs f_{upper} such that

(i) $f(x) \leq f_{upper}(x)$ for all x.

(ii) $\Pr[f_{upper}(x) \neq f(x)] \leq \varepsilon$ for a uniform random x.

(iii) f_{upper} has size $(1/\varepsilon)^{O(w)}$.

Theorem (This work)

Assume the above conjecture is true. There is a constant $c > 1$ such that, for any w-set system $\mathcal{F}$ with $|\mathcal{F}| \geq (\log w)^{c \cdot w}$, it contains a 3-sunflower.

Regular set system contains disjoint sets

Theorem

Assume the upper bound DNF sparsification conjecture holds. Then exists a constant $c > 0$, such that for any w-set system $\mathcal{F}$, if it is $(\log w)^c$-regular, then

$$\Pr_W[\exists S \in F, S \subseteq W] \geq 0.99$$

equivalently, $\Pr_x[f_{\mathcal{F}}(x) = 1] \geq 0.99]$

Regular set system contains disjoint sets

Theorem

Assume the upper bound DNF sparsification conjecture holds. Then exists a constant $c > 0$*, such that for any* w*-set system* $\mathcal{F}$*, if it is* $(\log w)^c$*-regular, then*

$$\Pr_W[\exists S \in F, S \subseteq W] \geq 0.99$$

equivalently, $\Pr_x[f_{\mathcal{F}}(x) = 1] \geq 0.99]$

Corollary

Assume the above conjecture holds. Then for any $(\log w)^c$*-regular set system* $\mathcal{F}$*, it contains two disjoint sets.*

From two disjoint sets to r disjoint sets.

Lemma

Assume the upper bound DNF sparsification conjecture holds. Then exists a constant $c > 0$, such that for any w-set system $\mathcal{F}$, if it is $(\log w)^{c \cdot r}$-regular, then $\mathcal{F}$ contains r disjoint sets.

From two disjoint sets to r disjoint sets.

Lemma

Assume the upper bound DNF sparsification conjecture holds. Then exists a constant $c > 0$, such that for any w-set system $\mathcal{F}$, if it is $(\log w)^{c \cdot r}$-regular, then $\mathcal{F}$ contains r disjoint sets.

Theorem

Assume the upper bound DNF sparsification conjecture holds. For any w-set system $\mathcal{F}$ with $|\mathcal{F}| \geq (\log w)^{c \cdot r \cdot w}$, it contains a r-sunflower

Open problems

Open problems

Question

Could we confirm the upper bound DNF sparsification conjecture?

Open problems

Question

Could we confirm the upper bound DNF sparsification conjecture?

Highlight conjecture

There is a constant c. For any $(\log w)^c$-regular w-set system $\mathcal{F}$,

$$\Pr_x[f_{\mathcal{F}}(x) = 1] \geq 0.99$$

Open problems

Question

Could we confirm the upper bound DNF sparsification conjecture?

Highlight conjecture

There is a constant c. For any $(\log w)^c$-regular w-set system $\mathcal{F}$,

$$\Pr_x[f_{\mathcal{F}}(x) = 1] \geq 0.99$$

I guess for each variable x_i, the $I_{f_{\mathcal{F}}}(x_i) \ll 1/w$.

Thanks!