Imperfect Gaps in Gap-ETH and PCPs

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Introduction

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- Hardness/Easiness of finding approximate solutions to satisfiable CSPs as compared to unsatisfiable ones?
- Is it easier to build PCPs with imperfect completeness as compared to perfect completeness?

Gap-ETH and Perfect Completeness

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We will also refer to this as Gap-k-CSP.

For this presentation, we will think of a Gap-CSPs on n variables and $m = O(n)$ clauses.

Our Problems

Problem (1)

Is MAX 3-SAT$(1, .98)$ *"easier" than MAX 3-SAT*$(.99, .97)$*?*

The Gap-ETH Conjecture

Conjecture (Gap-ETH(Dinur'16 and MR'17))

For some constant $\tau > 0$, MAX 3-SAT$(1, 1 - \tau)$ does not have a $2^{o(n)}$ randomized algorithm.

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Conjecture (Gap-ETH without perfect completeness)

For some constants $\epsilon > \gamma > 0$, MAX 3-SAT$(1-\gamma, 1-\epsilon)$ does not have a $2^{o(n)}$ randomized algorithm.

Equivalence of Gap-ETH conjectures

Theorem

The Gap-ETH conjecture is equivalent to the Gap-ETH conjecture without perfect completeness i.e.
For all constants $\tau > 0$*, MAX 3-SAT*$(1, 1-\tau)$ *has a* $2^{o(n)}$ *time algorithm* $\iff$ *for all constants* $\epsilon > \gamma > 0$*, MAX 3-SAT*$(1-\gamma, 1-\epsilon)$ *has a* $2^{o(n)}$ *time algorithm.*

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We will present:

Theorem

If for all constants $\tau > 0$, *MAX 3-SAT*$(1, 1 - \tau)$ *has a* $2^{o(n)}$ *time randomized algorithm, then for all constants* $\delta > 0$, *MAX 3-SAT*$(.99, .97)$ *has a* $2^{\delta n}$ *time randomized algorithm.*

Proof Sketch

Lemma

For large enough constant k, there exists a randomized reduction from MAX 3-SAT$(.99, .97)$ on n variables and $O(n)$ clauses to MAX $3k$-CSP$(1, 1/2)$ on n variables and $O(n)$ clauses, such that:

- *YES instances reduce to YES instances with probability $\geq 2^{-n/k}$.*
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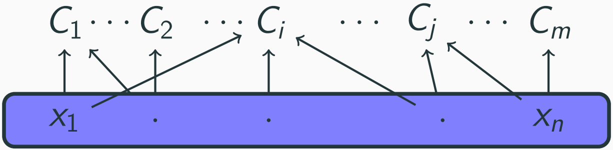

frac of $1's > .99$

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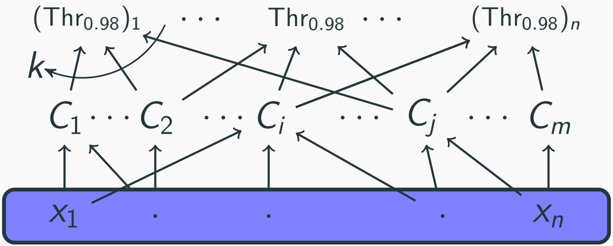

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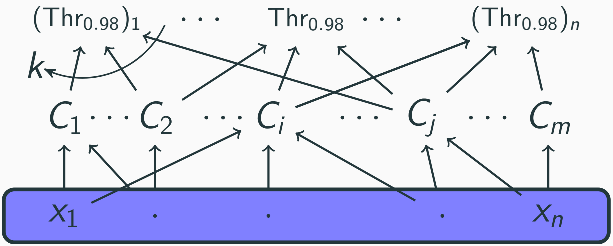

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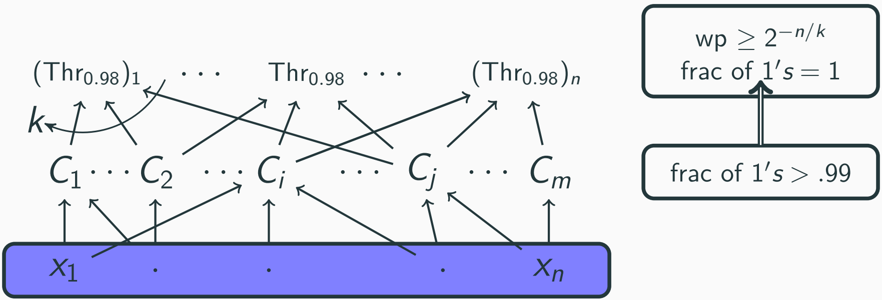

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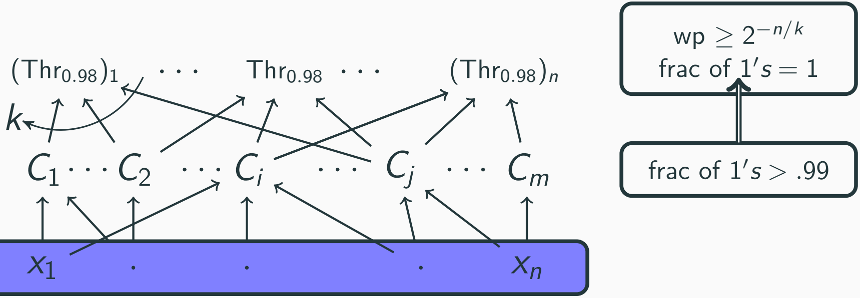

Note that this gives us a $3k$-CSP.

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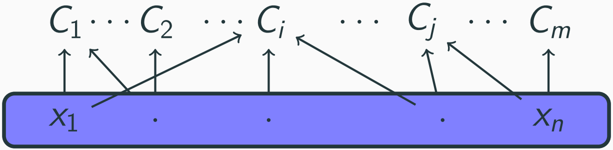

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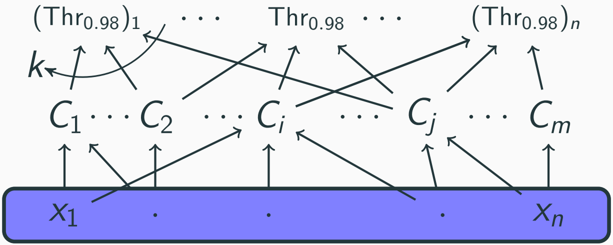

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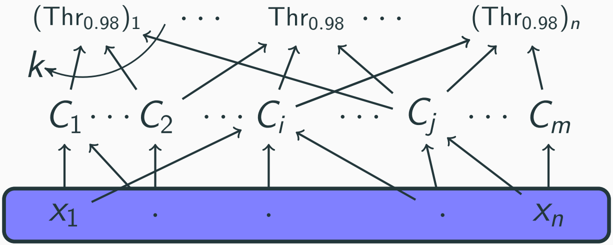

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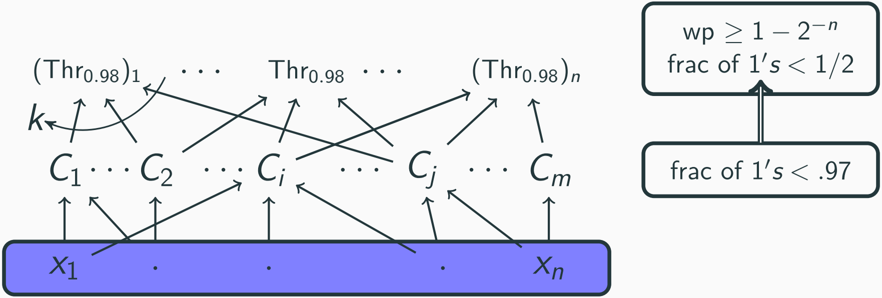

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- Run the $2^{o(n')}$ algorithm on the MAX 3-SAT$(1, 1 - \Omega_k(1))$ instances and output YES if the algorithm outputs YES on any of the produced instances.
- Total running time $2^{n/k}n^2 \cdot 2^{o(n')} = 2^{n/k + o(n)} \leq 2^{\delta n}$ for large enough constant k.

Derandomization using samplers

- One-sided derandomization using samplers. We use LLL to handle the completeness case.

PCPs and Perfect Completeness

Definition of PCPs

$\mathrm{PCP}_{c,s}[r, q]$ with proof size n:

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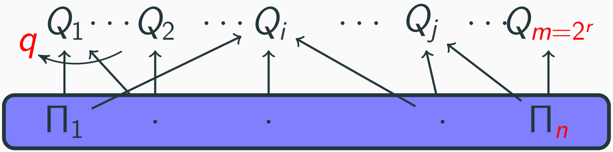

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- Linear-sized PCP with long queries [BKKMS'13]:

$$NTIME[O(n)] \subseteq \mathrm{PCP}_{1,1/2}[\log n + O_\epsilon(1), n^\epsilon],$$

with a $O_\epsilon(n)$ proof size.

Linear-Sized PCP conjecture

Conjecture (Linear-sized PCP conjecture)

$NTIME[O(n)]$ has linear-sized PCPs, i.e.
$NTIME[O(n)] \subseteq PCP_{1,s}[\log n + O(1), O(1)]$ for some constant $s < 1$.

Our Question

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- Can we convert an imperfect PCP to a perfect completeness PCP in a blackbox manner?

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- One can just apply the best known PCPs for NTIME$[O(n)]$, for example
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- Bellare Goldreich and Sudan [1] studied many such black-box reductions between PCP classes. Their result for transferring the gap to 1:

$$\mathrm{PCP}_{c,s}[r, q] \leq_R \mathrm{PCP}_{1,rs/c}[r, qr/c].$$

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We show a blackbox way to transfer a PCP with imperfect completeness to one with perfect completeness, while incurring a small loss in the query complexity, but maintaining other parameters of the original PCP.

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We will show how to build a new proof system (specify proof bits and verifier queries) for L that has completeness 1 and soundness < 1.

A Robust Circuit using Thresholds

Π_1 · · · Π_n

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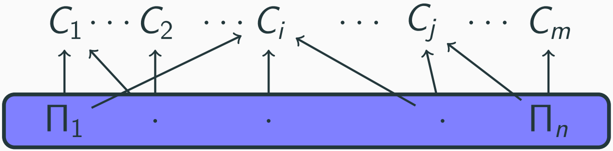

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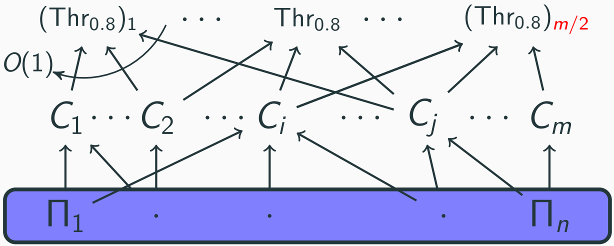

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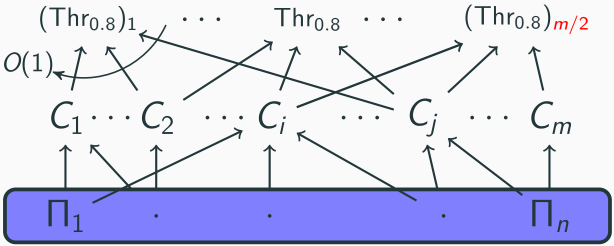

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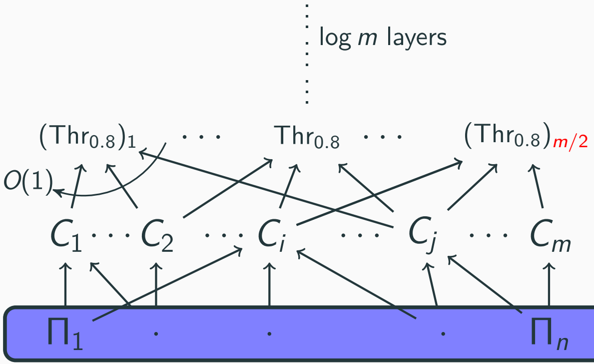

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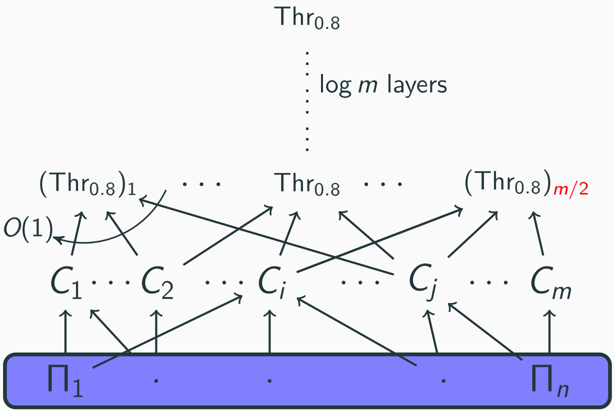

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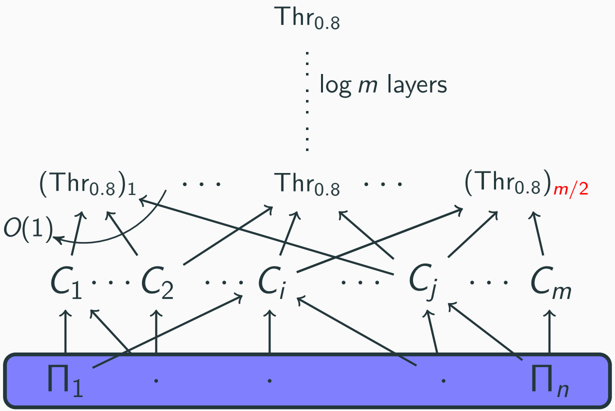
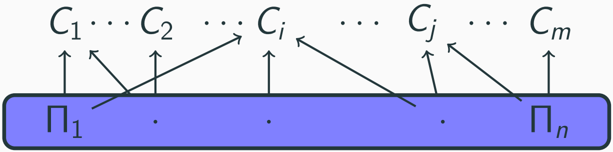


We can derandomize this using samplers.

Increasing fraction of 1's

$\Pi_1 \quad \cdot \quad \cdot \quad \cdot \quad \Pi_n$

Increasing fraction of 1's



frac of $0's < .1$

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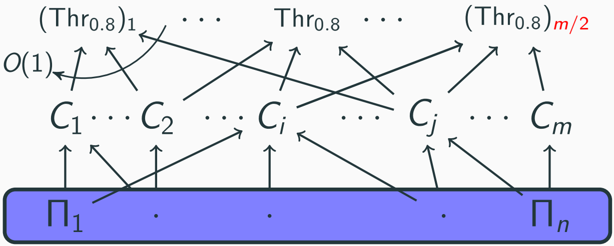

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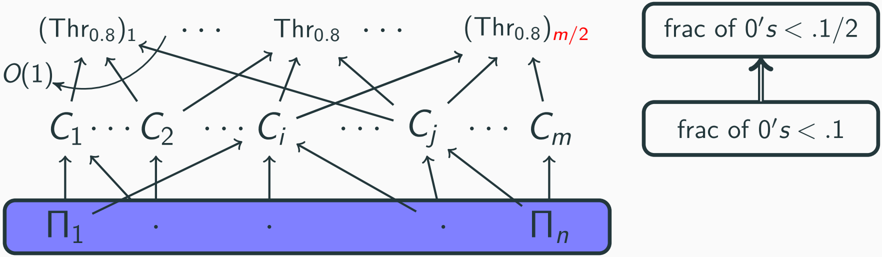

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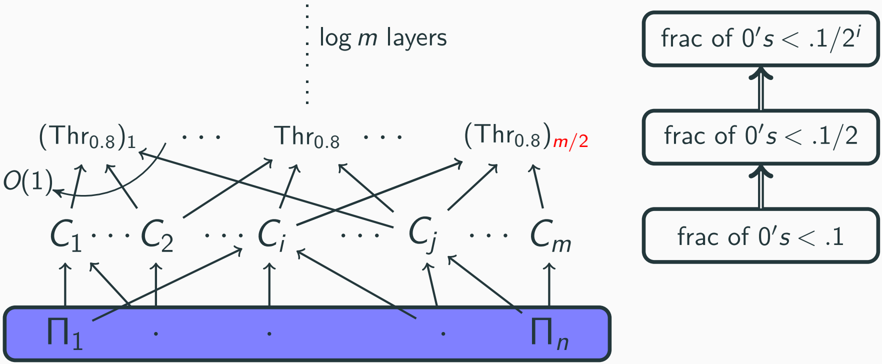

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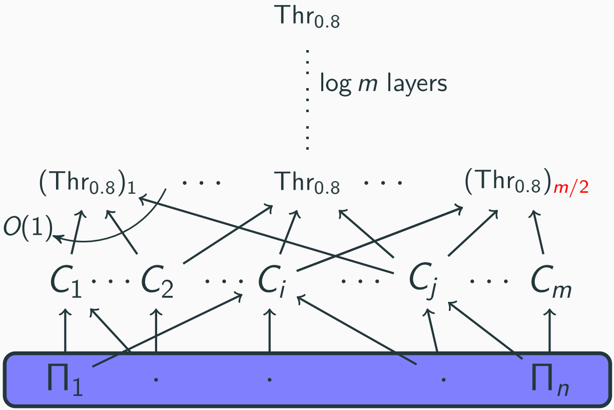

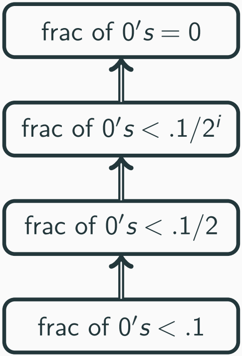

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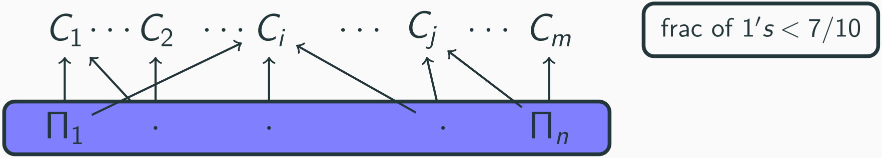

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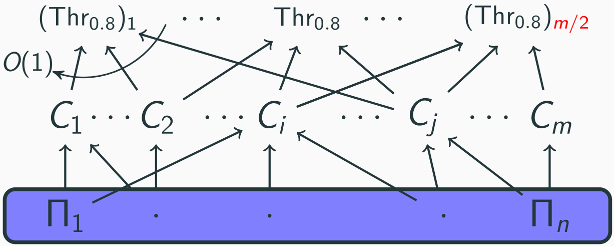
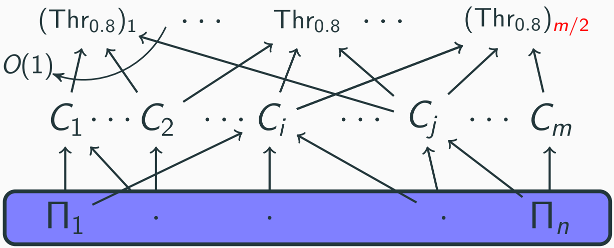
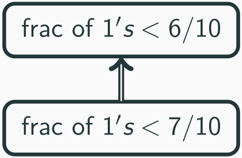


frac of $1's < 7/10$

Maintaining fraction of 1's





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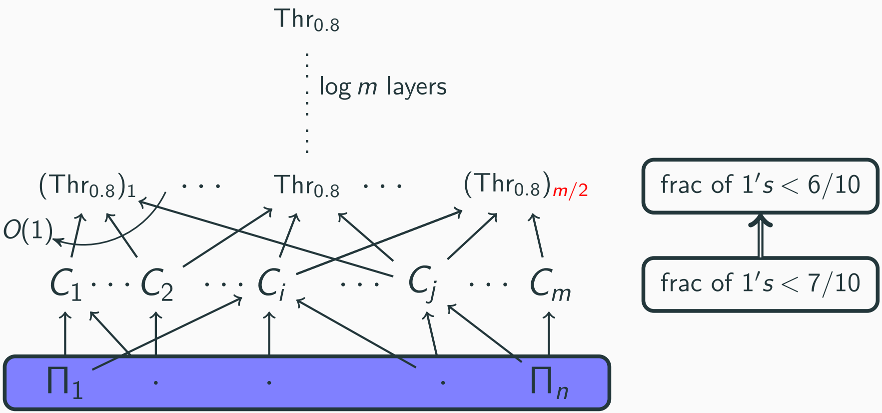

Final PCP

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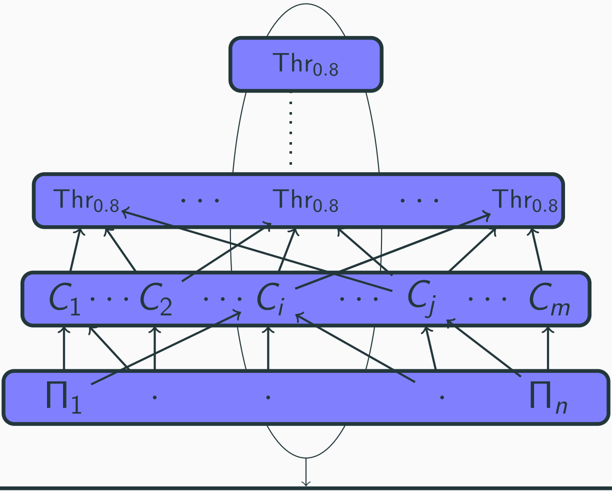

In a single query, we will verify all included gates:
check whether each gate's output is consistent with its inputs
and the top gate evaluates to 1

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- Queries: $q + O(\log m) = q + O(r)$
- Randomness complexity: r (stays the same)
- Size: $O(m)$

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Comparison to Best-Known PCPs

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We get the following result for $NTIME[O(n)]$:

Corollary

For all constants, c, s, s', if $\mathrm{NTIME}[O(n)] \subseteq \mathrm{PCP}_{c,s}[\log n + O(1), q]$, then $\mathrm{NTIME}[O(n)] \subseteq \mathrm{PCP}_{1,s'}[\log n + O(1), q + O(\log n)]$.

Comparison to Best-Known PCPs

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While the current best known linear-sized PCP is:

$$NTIME[O(n)] \subseteq \mathrm{PCP}_{1,s}[\log n + O_\epsilon(1), n^\epsilon],$$

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- Our results imply that building linear-sized PCPs with minimal queries for NTIME[$O(n)$] and perfect completeness should be nearly as hard (or easy!) as linear-sized PCPs with minimal queries for NTIME[$O(n)$] and imperfect completeness.
- We show the equivalence of Gap-ETH under perfect and imperfect completeness, i.e. Max-3SAT with perfect completeness has $2^{o(n)}$ randomized algorithms iff Max-3SAT with imperfect completeness has $2^{o(n)}$ algorithms.

Open Problems

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- Can we derandomize the reduction from Gap-ETH without perfect completeness to Gap-ETH?
- Blackbox reductions to get better parameters for MAX k-CSP? Currently we know that MAX k-CSP$(1, 2^{O(k^{1/3})}/2^k)$ for satisfiable instances whereas for unsatisfiable instances MAX k-CSP$(1-\epsilon, 2k/2^k)$ (which is tight up to constant factors).

Thanks! Questions?

References i

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