A Fine-Grained Analogue of Schaefer's Theorem in P: Dichotomy of $\exists^k\forall$-Quantified First-Order Graph Properties

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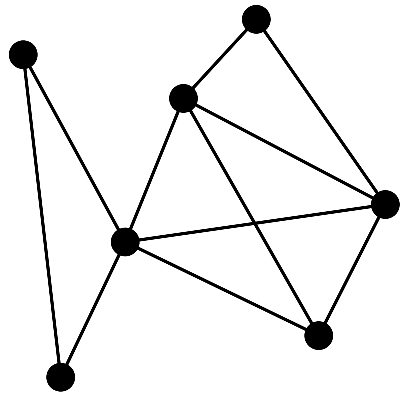

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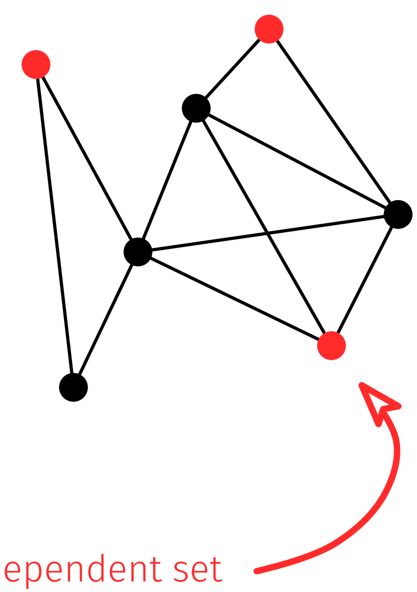

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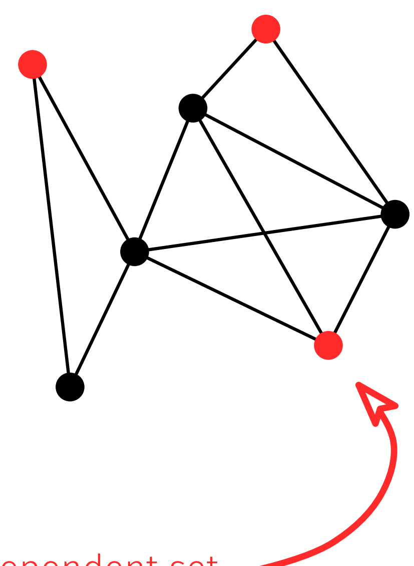

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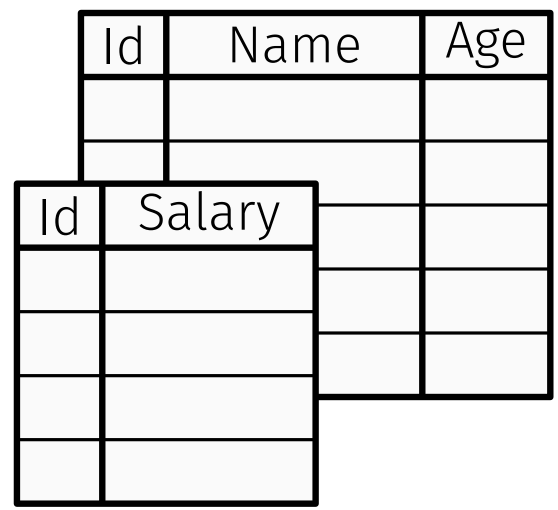

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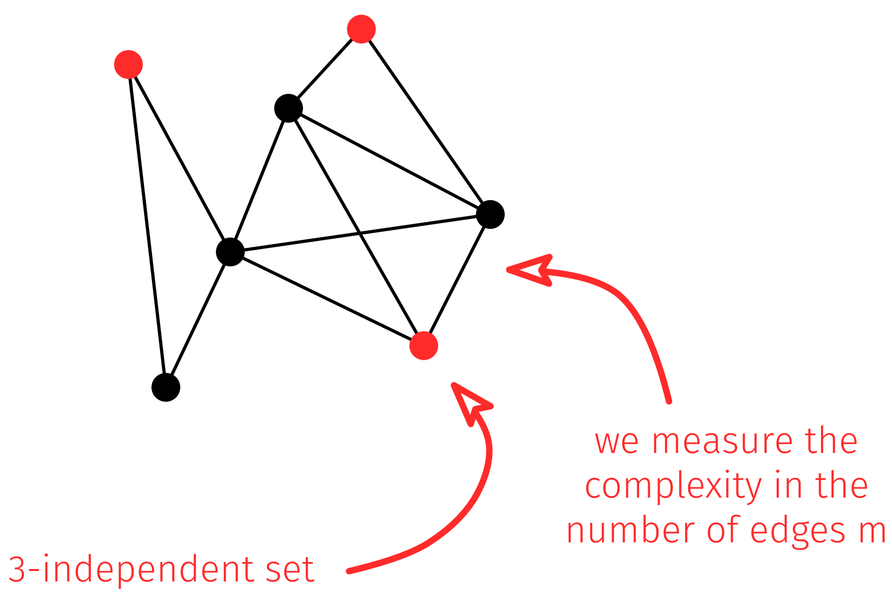

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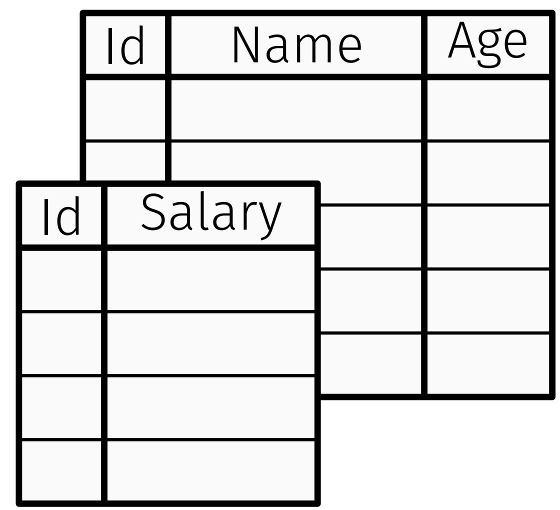

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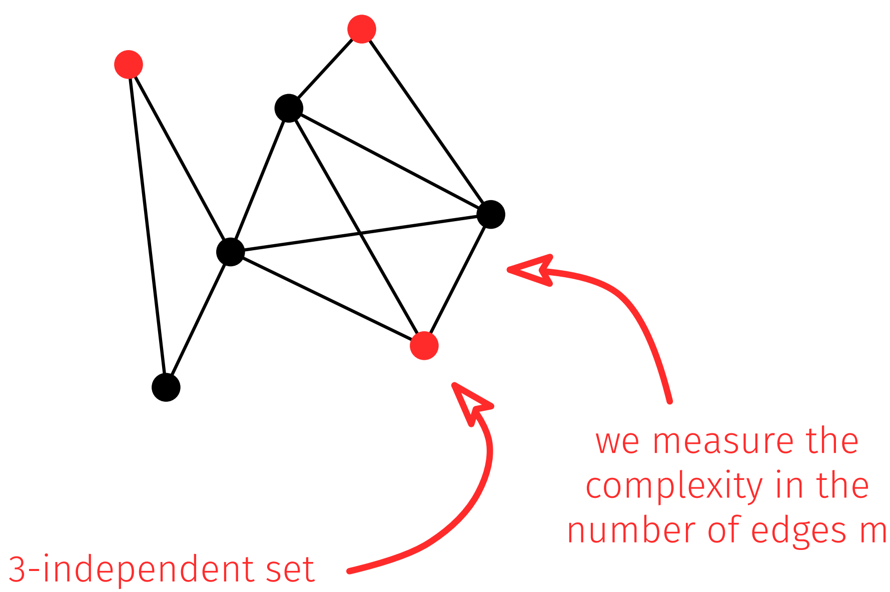

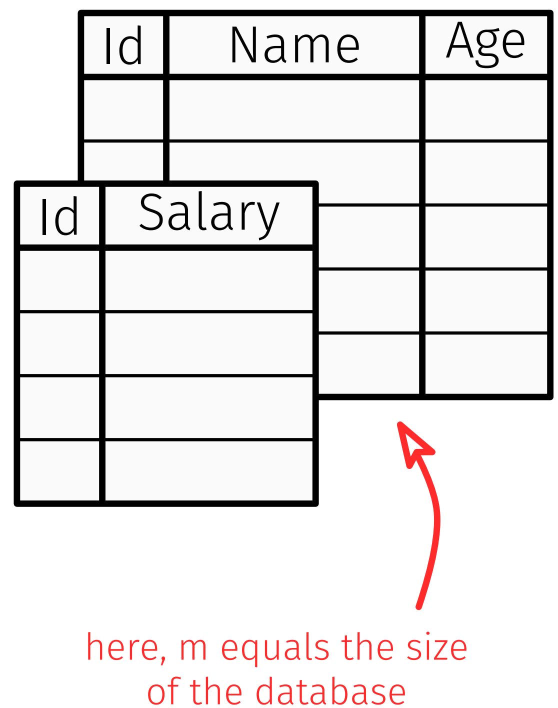

Orthogonal Vectors

Given two sets $X_1, X_2 \subseteq \{0,1\}^d$ of size n, check whether there exists an orthogonal pair $x_1 \in X_1, x_2 \in X_2$

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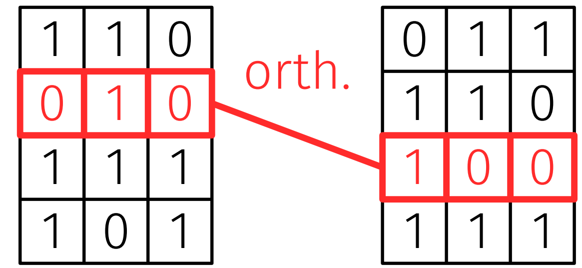
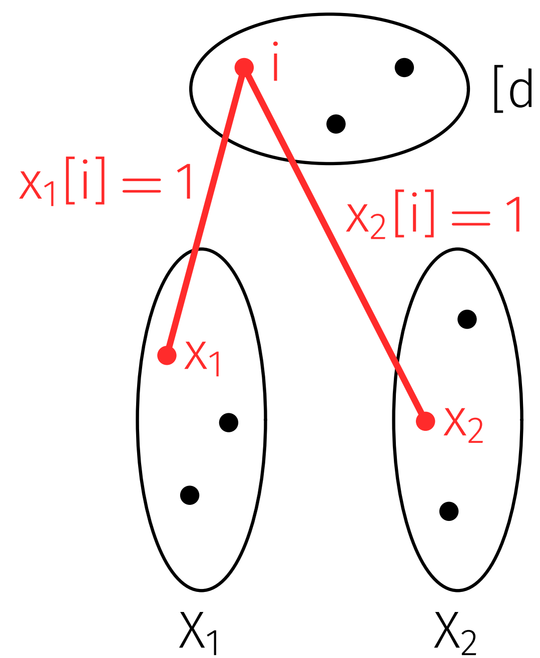


$$\exists x_1 \in X_1 \, \exists x_2 \in X_2 \, \forall i \in [d]: \\ x_1[i] = 0 \lor x_2[i] = 0$$



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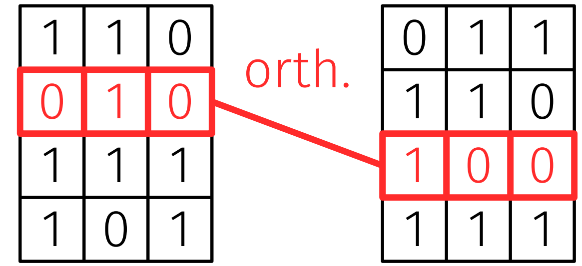

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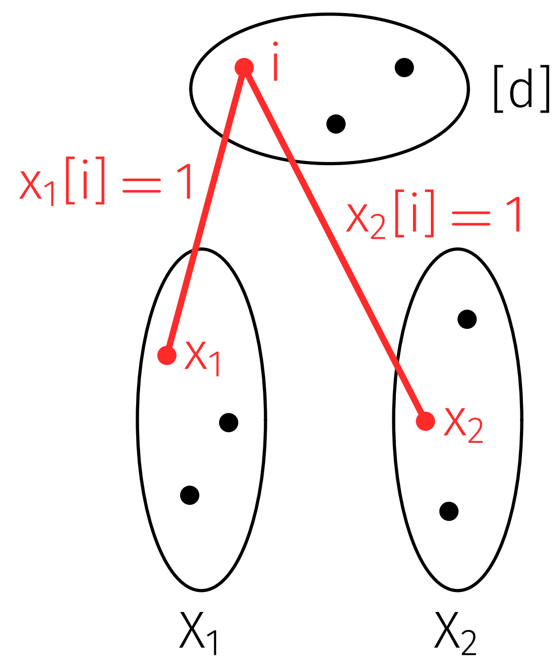

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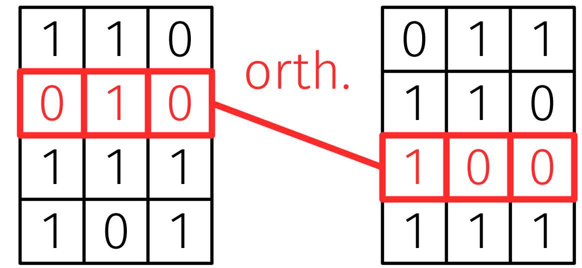

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$$\exists x_1 \in X_1 \;\ldots\; \exists x_k \in X_k \; \forall i \in [d]:$$
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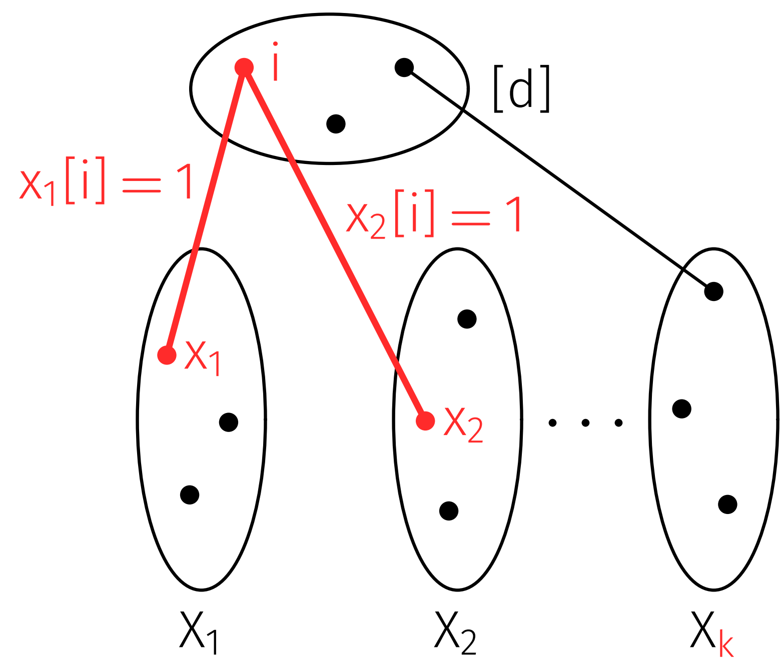

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$$O(m^k) \ \ \textit{vs.} \ \ O(m^{k-0.01})$$

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Constraint satisfaction problems	First-order properties
3-SAT is NP-complete [Cook '71]	k-OV is FOP-complete [GIKW '17]
Every Boolean CSP is either in P or NP-complete [Schaefer '78]	?

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for any subset $J \subseteq [k]$ of $k - h$ inputs,

there exists an assignment $\alpha : J \to \{0,1\}$,

so that $\varphi|_\alpha$ has exactly one falsifying assignment

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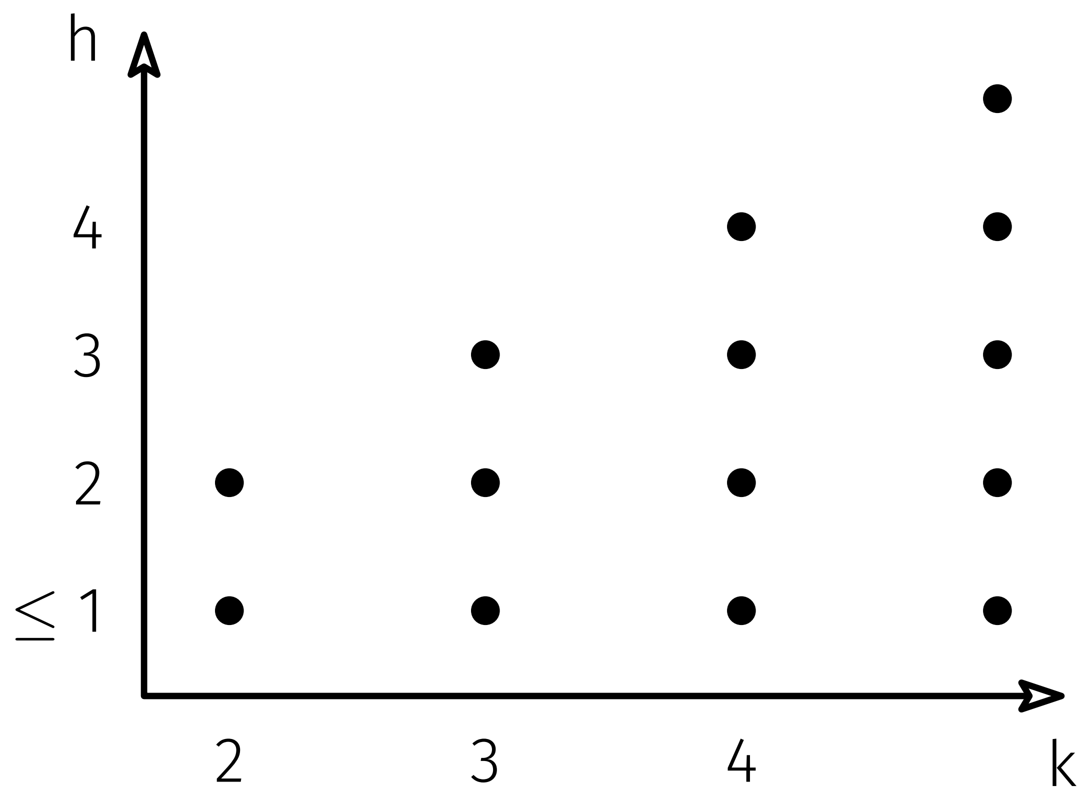

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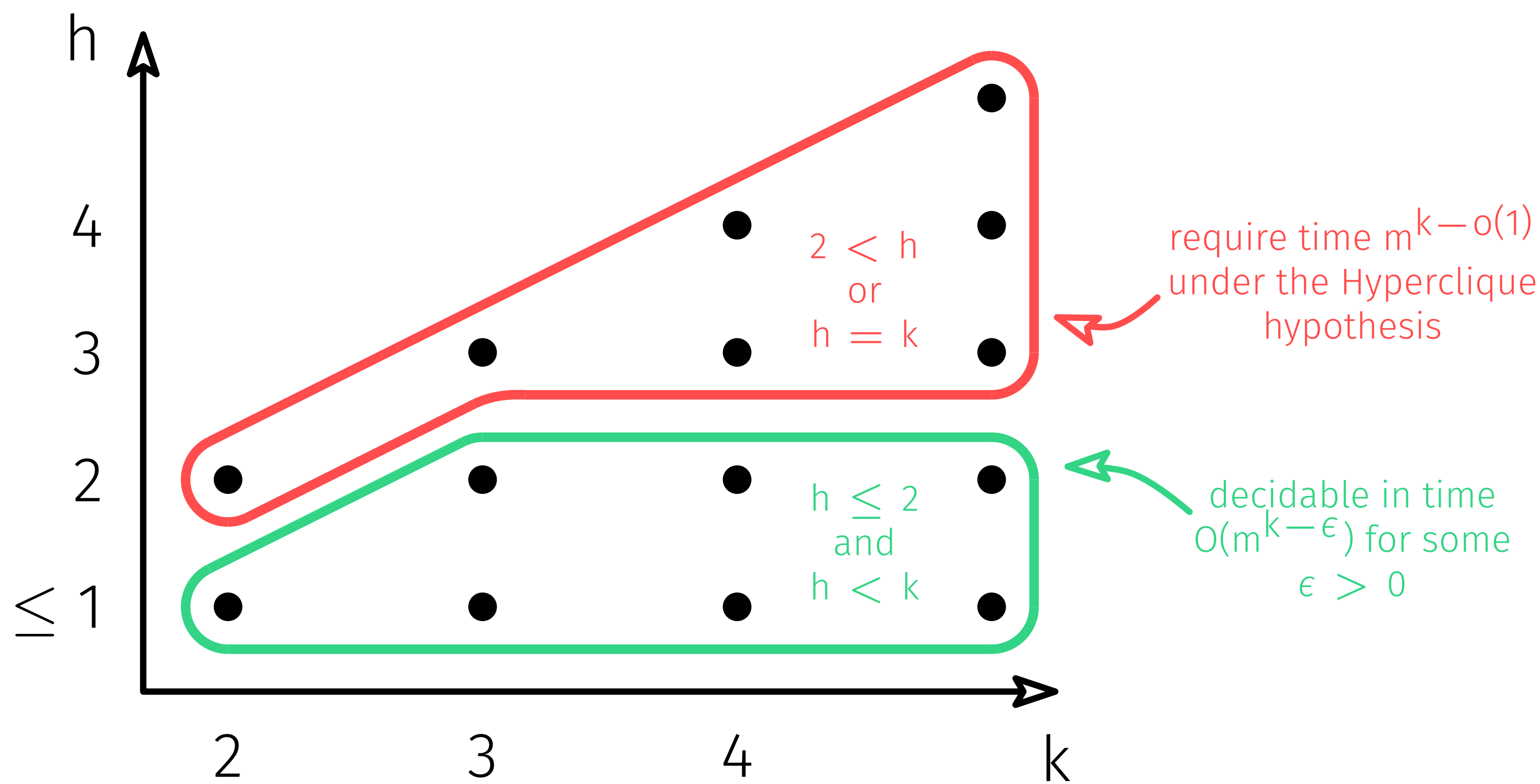

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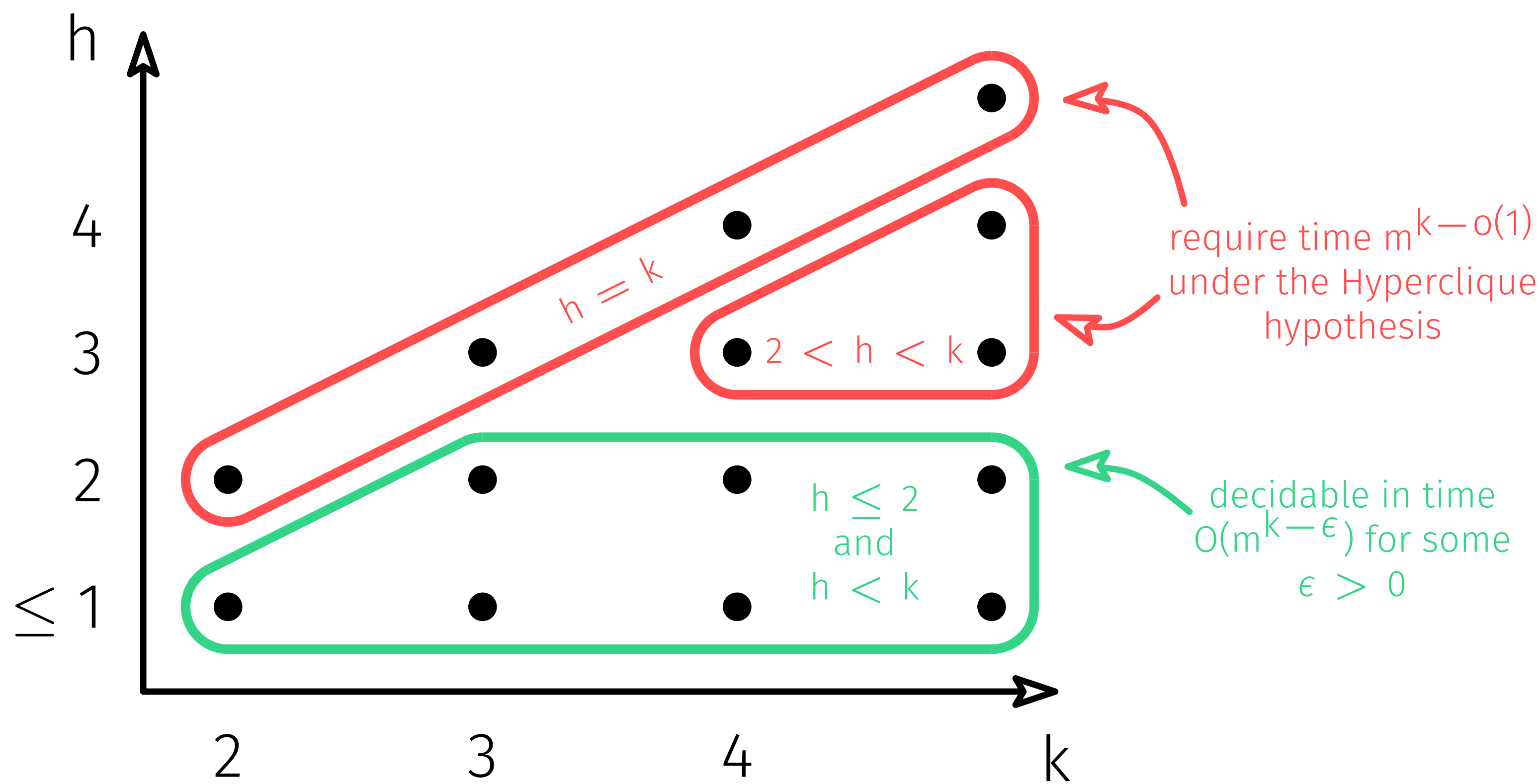

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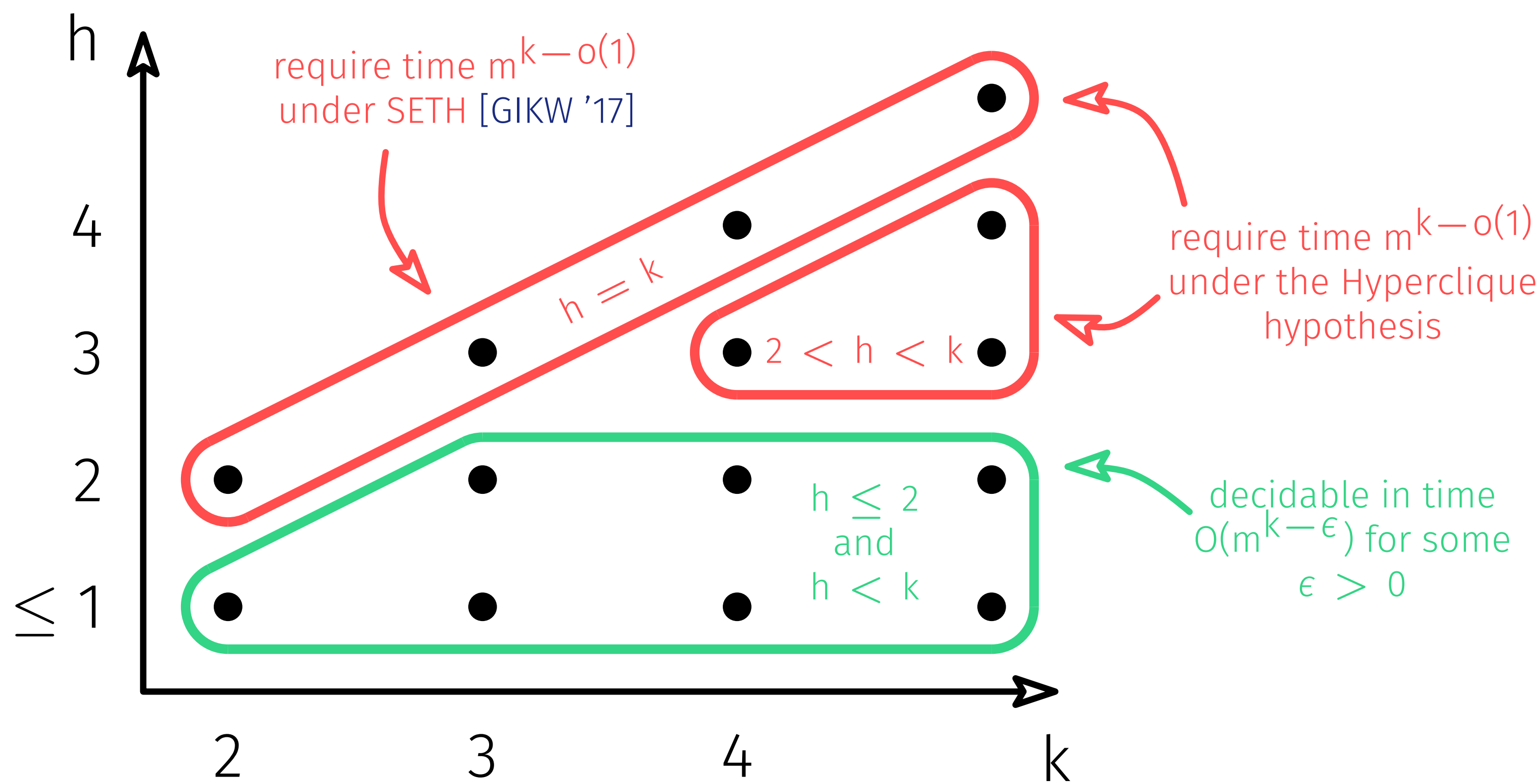

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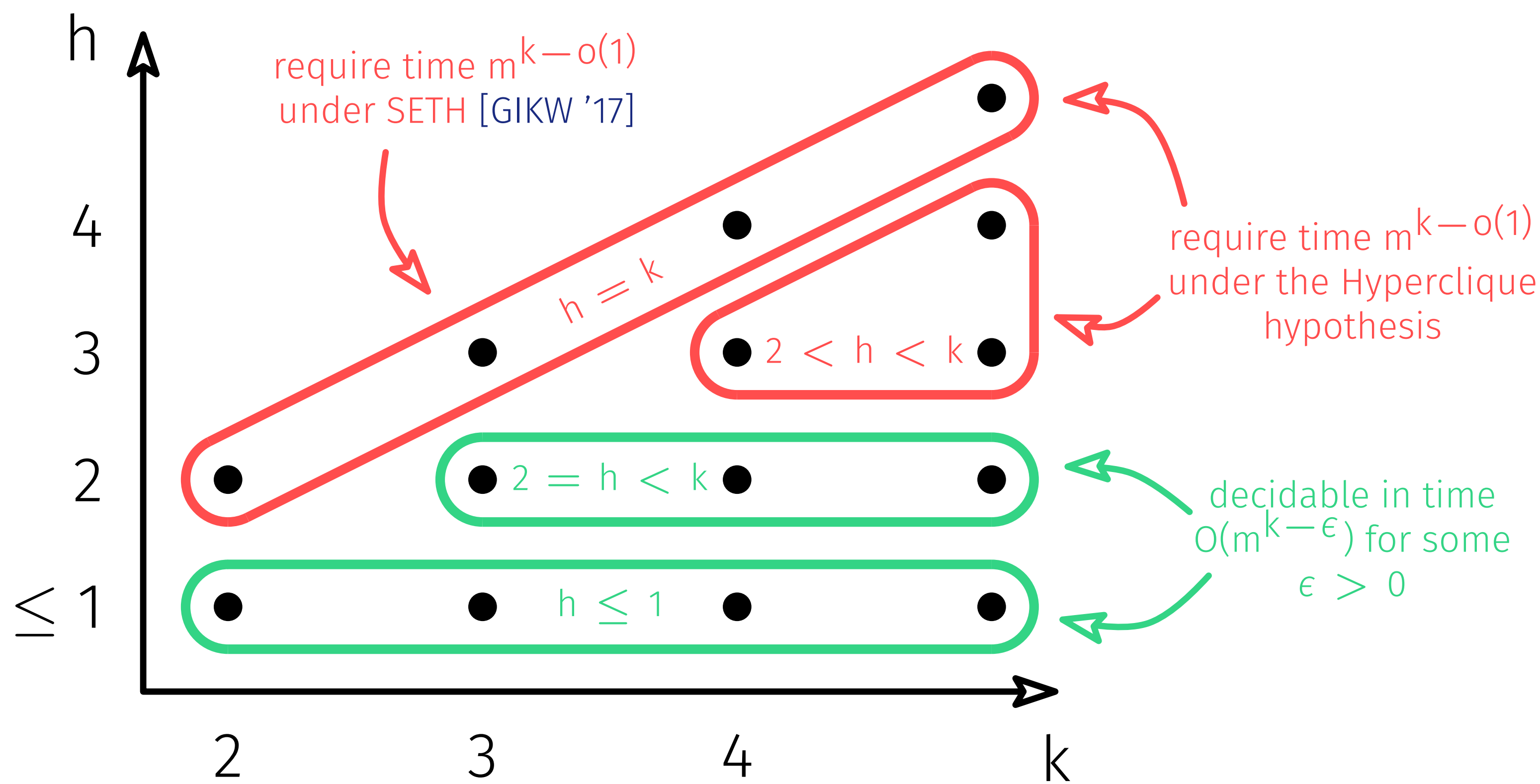

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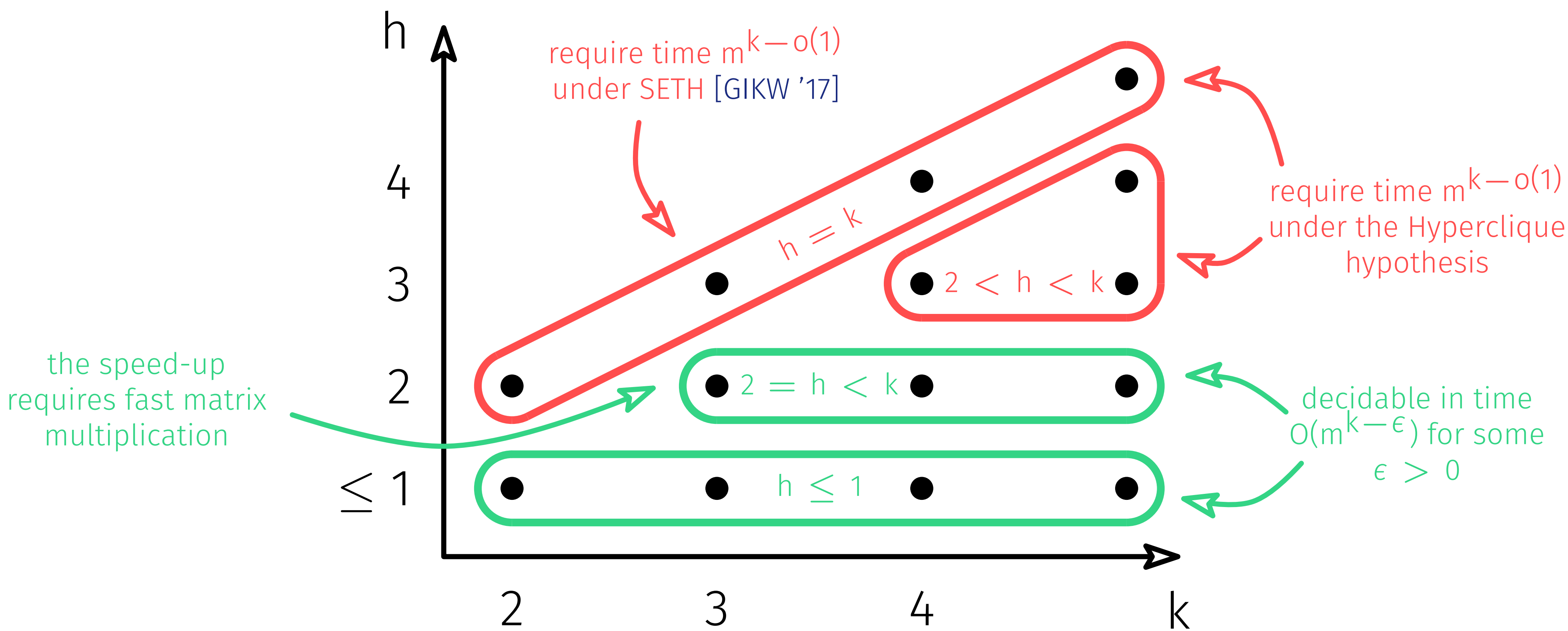

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Hypothesis: h-uniform Hyperclique

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Our results: Hardness levels

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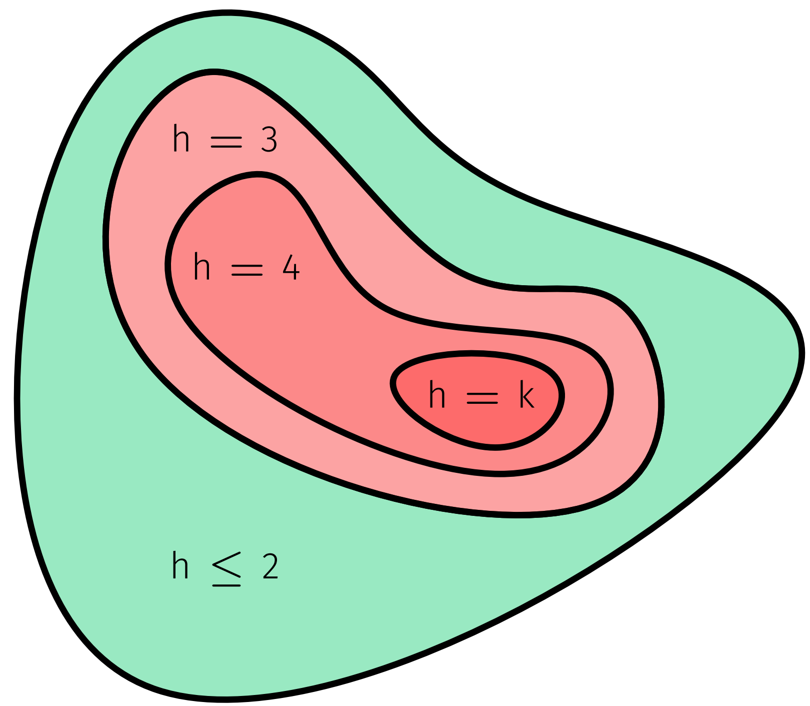

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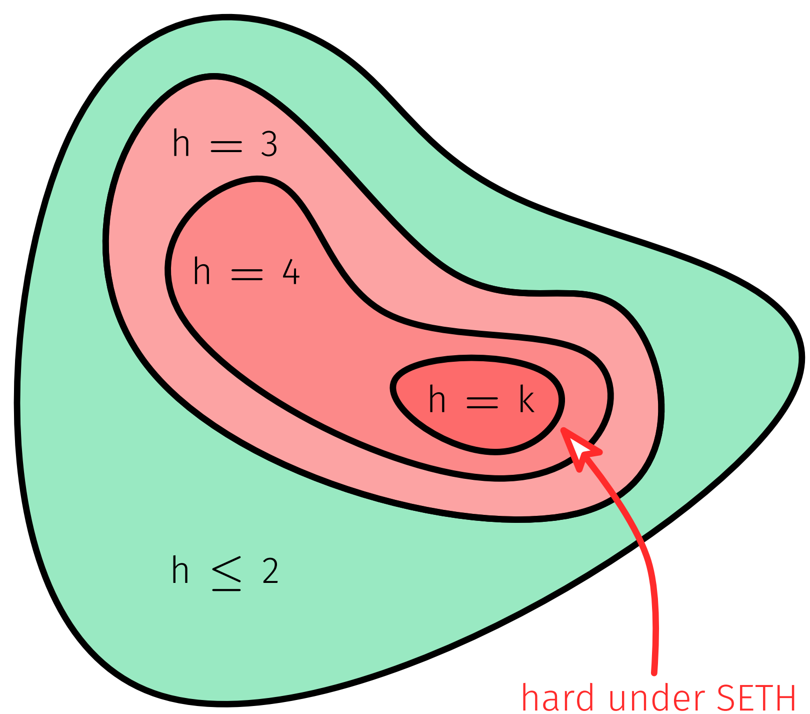
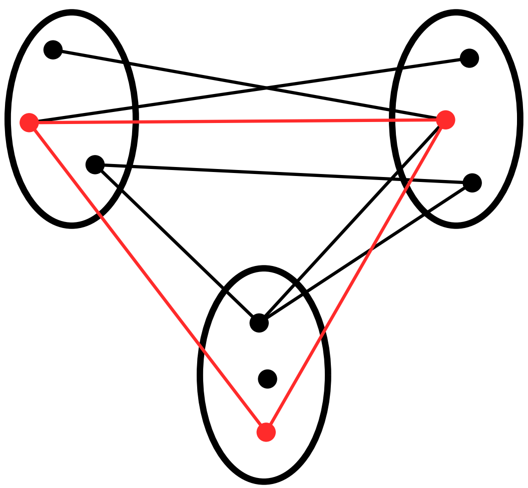


Build your own SUBcubic problem!

Step 1: Take the basis problem

$\cancel{O(n^3)}$ $O(n^{\omega})$



(Triangle Detection)

Step 2: Choose your toppings

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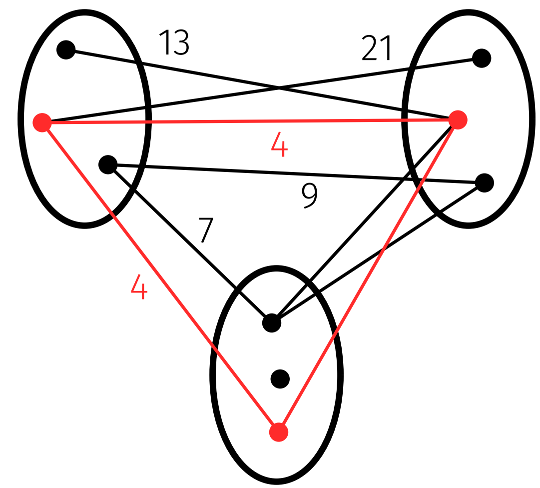
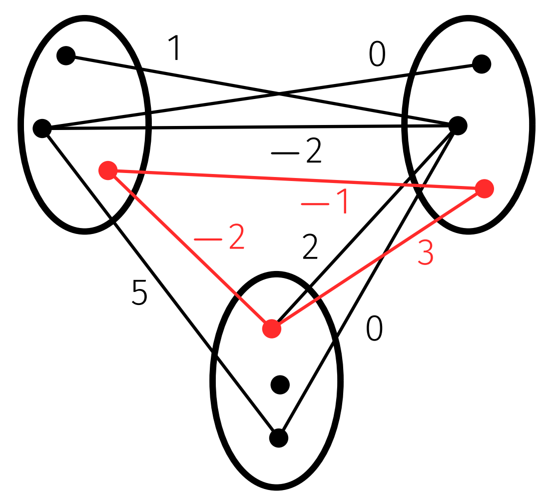


(Equal Constraint)



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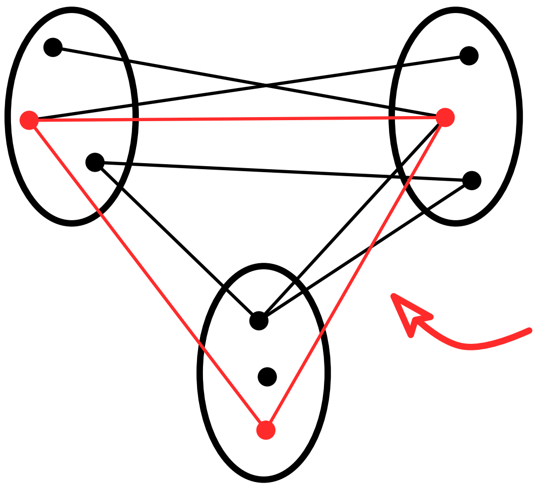

we assume tripartite graphs

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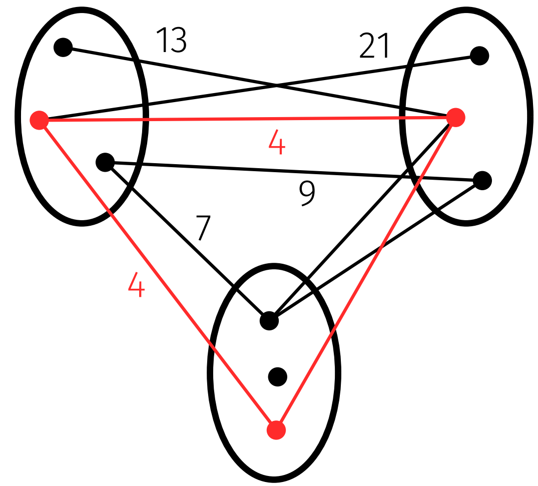

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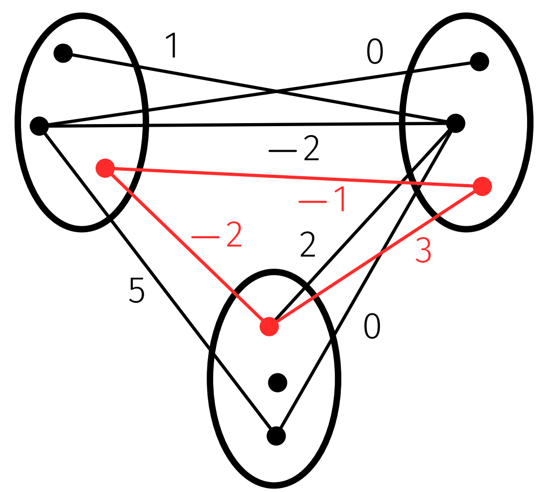

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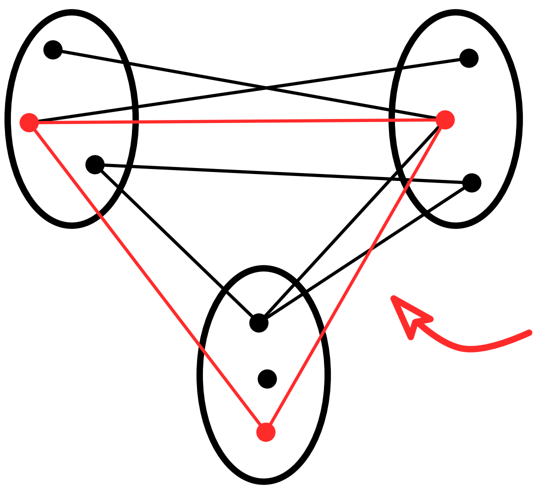

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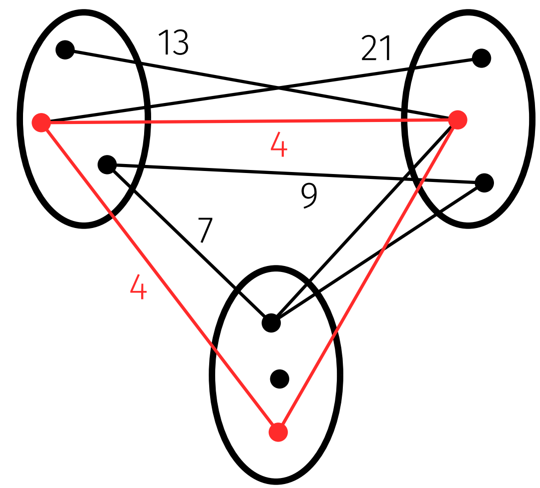

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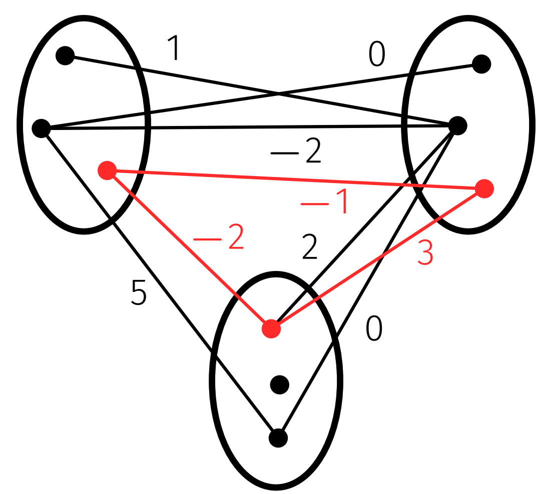
works for any target t
(here $t = 0$)



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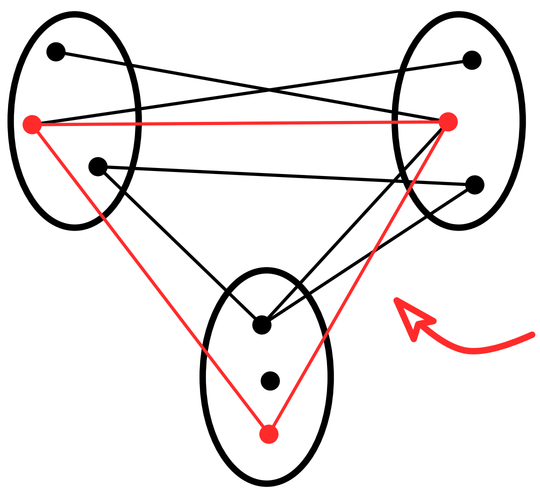

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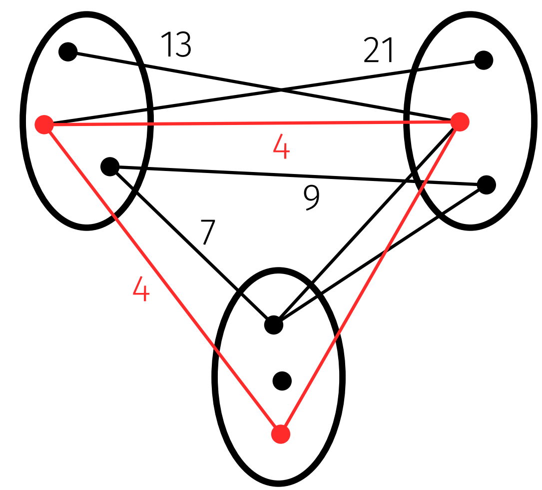

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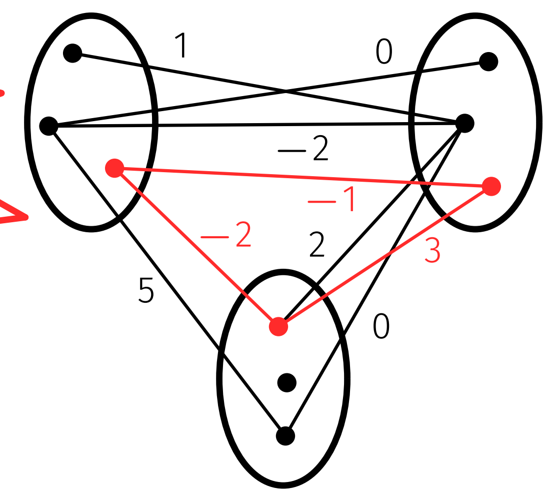
works for any target t (here $t = 0$)

small total weight: $\sum_e |w(e)| \leq O(n^2)$

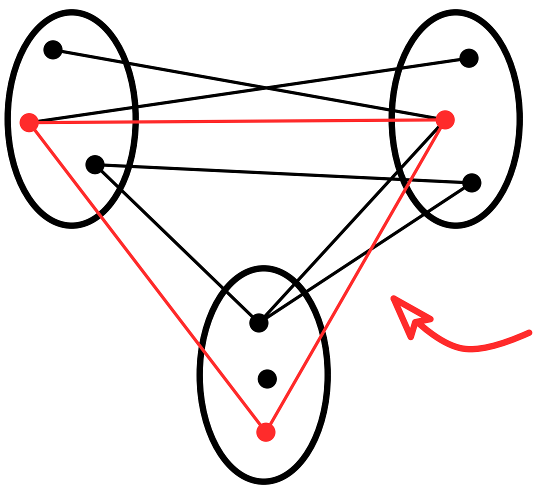


(Sum Constraint)

Build your own SUBcubic problem!

Step 1: Take the basis problem

$\cancel{O(n^3)}$ $O(n^{\omega})$

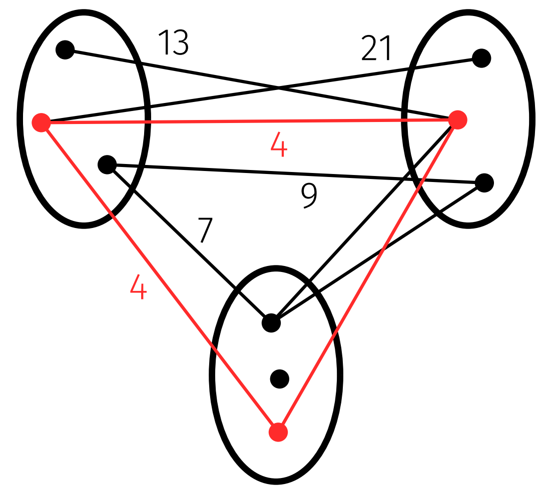


we assume tripartite graphs

(Triangle Detection)

Step 2: Choose your toppings

$\cancel{O(n^3)}$ $O(n^{3-\varepsilon})$



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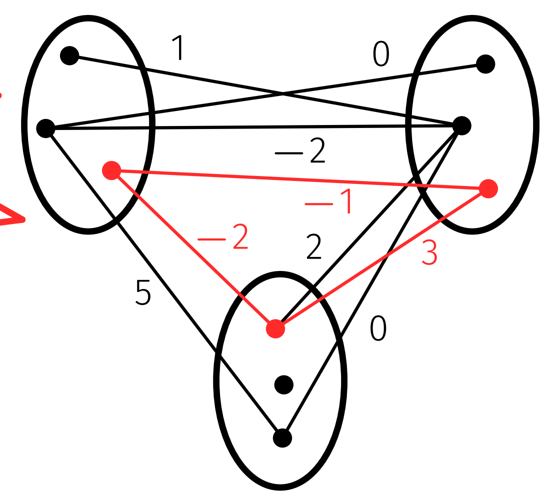
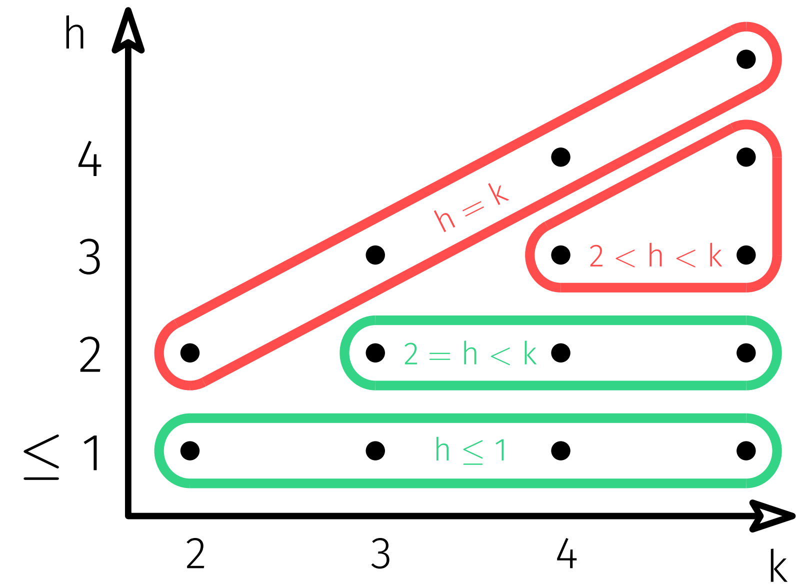


small total weight: $\sum_e |w(e)| \leq O(n^2)$

(Sum Constraint)

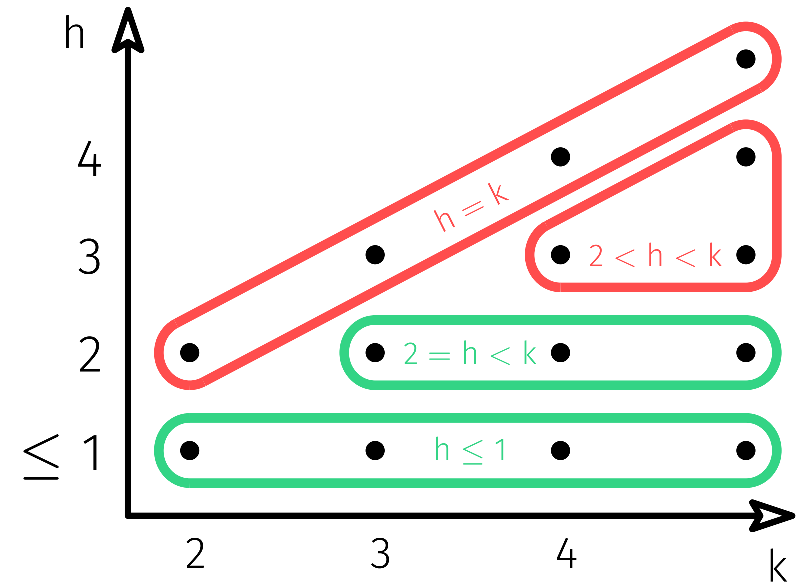
“Constrained Triangle Detection”

Algorithms for Properties of Hardness $h \leq 2$



Algorithms for Properties of Hardness $h \leq 2$

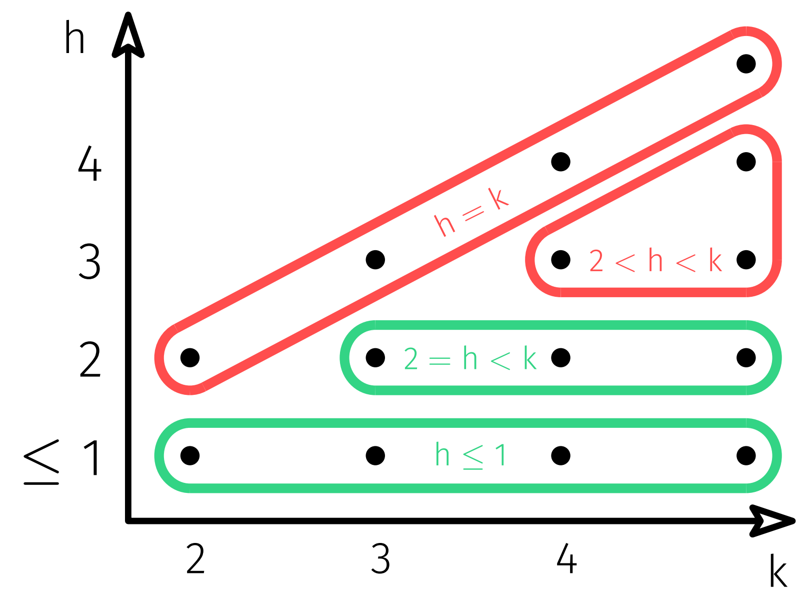
$k = 3$: Reduction to Constrained Triangles



Algorithms for Properties of Hardness $h \leq 2$

$k = 3$: Reduction to Constrained Triangles

$$\exists x_1 \exists x_2 \exists x_3 \forall y:$$
$$\varphi(E(x_1, y), E(x_2, y), E(x_3, y))$$

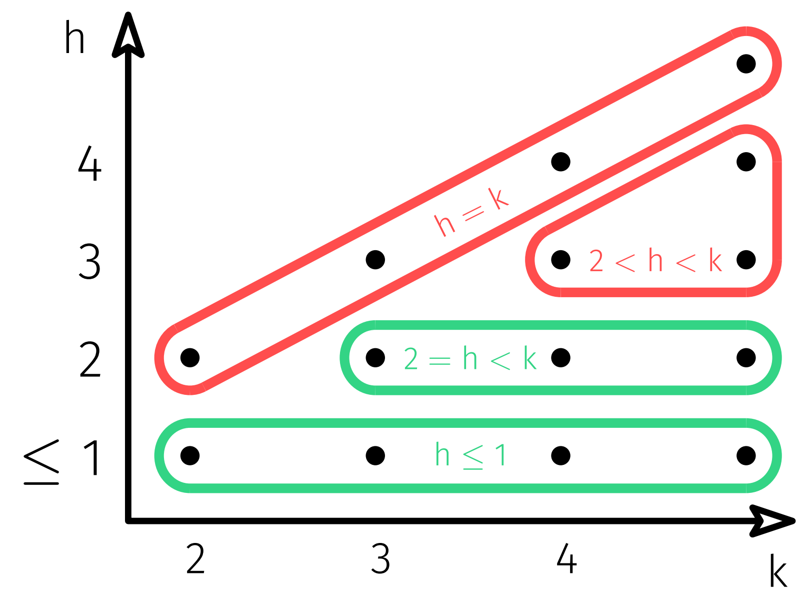


Algorithms for Properties of Hardness $h \leq 2$

$k = 3$: Reduction to Constrained Triangles

think of x_1, x_2, x_3 as vectors

$\exists x_1 \, \exists x_2 \, \exists x_3 \, \forall y$:
$\varphi(E(x_1, y), E(x_2, y), E(x_3, y))$



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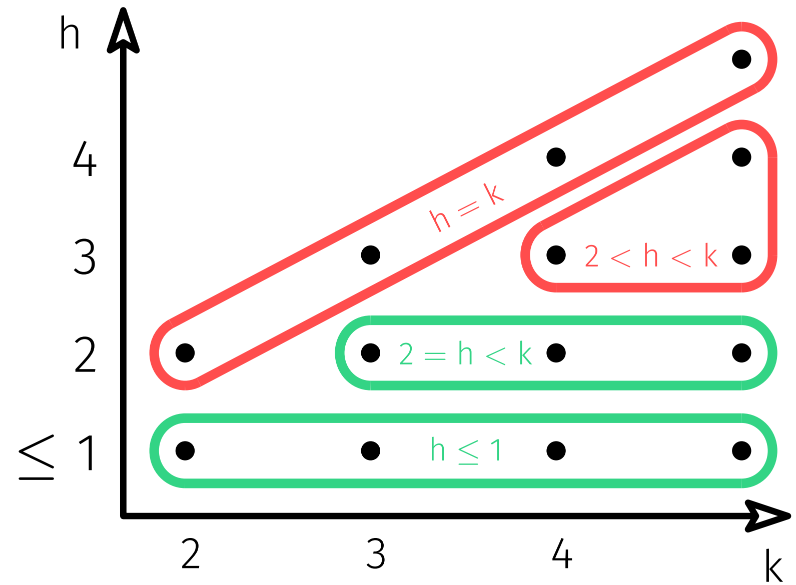

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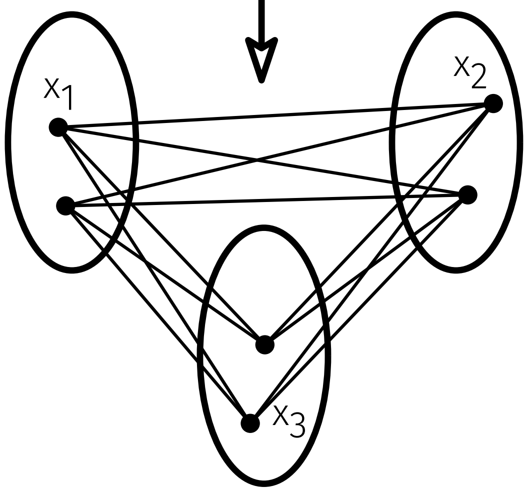

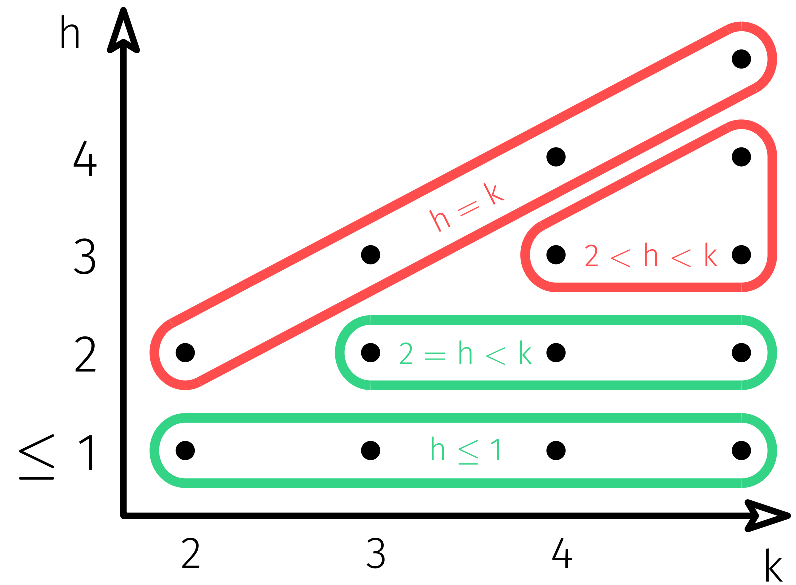

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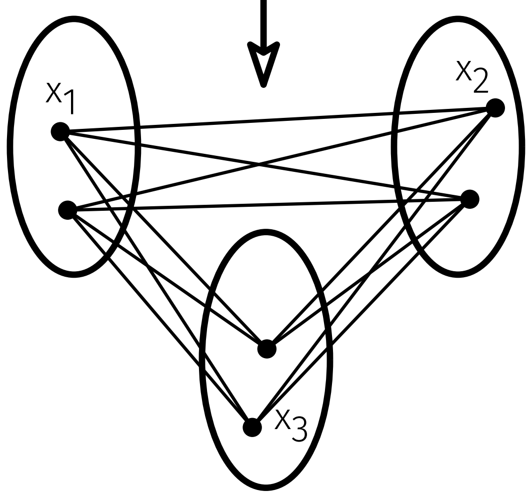
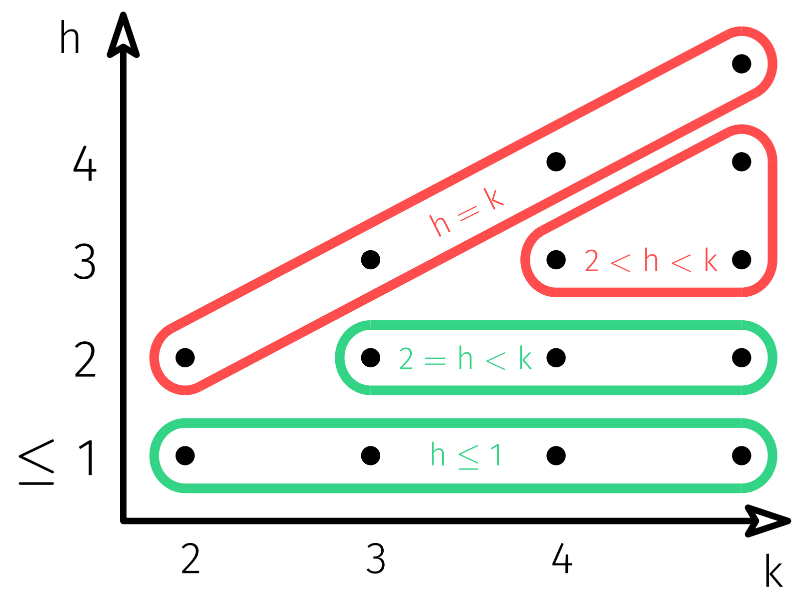


O(m) vertices



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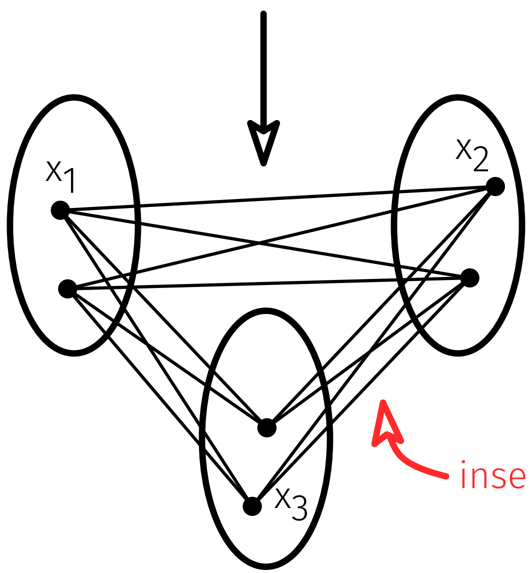
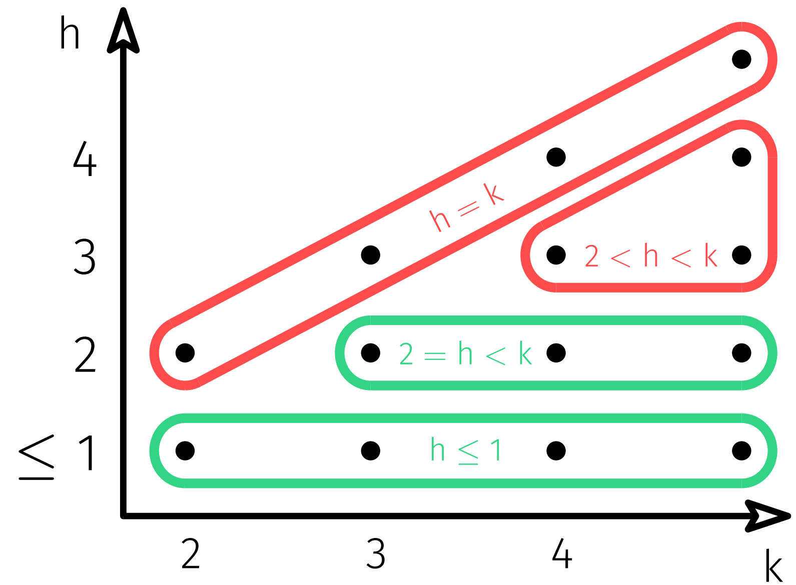


O(m) vertices

insert all edges



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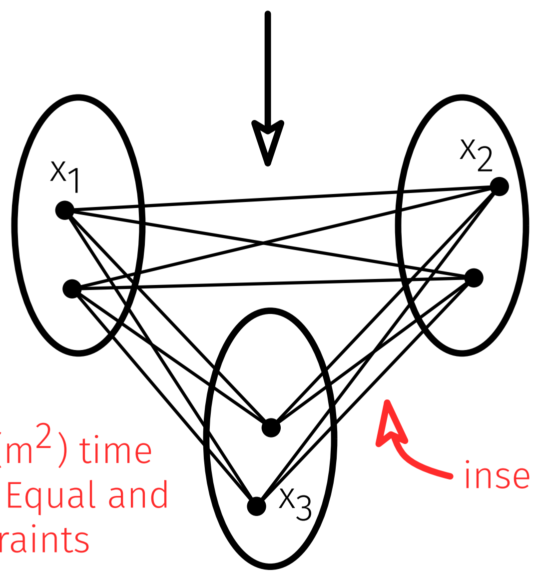
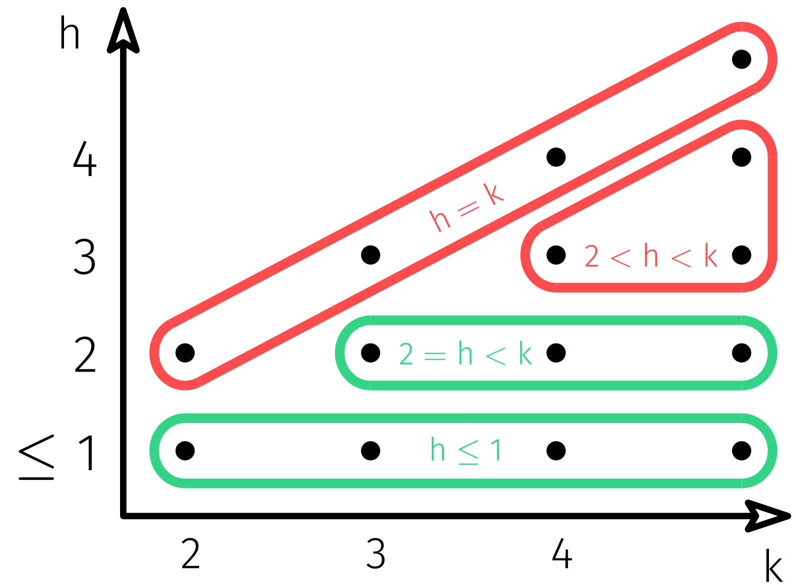


O(m) vertices

insert all edges

Idea: spend $O(m^2)$ time to encode φ by Equal and Sum constraints



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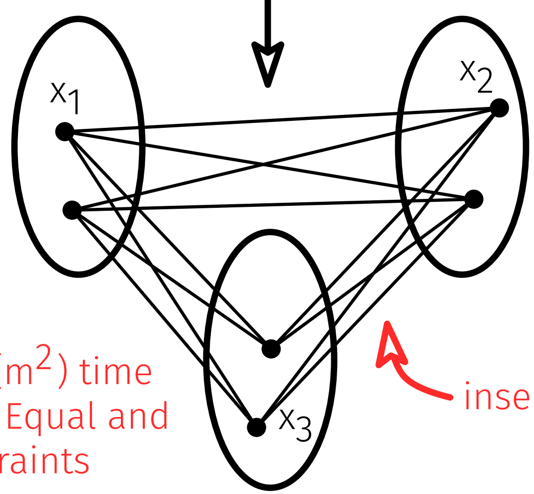

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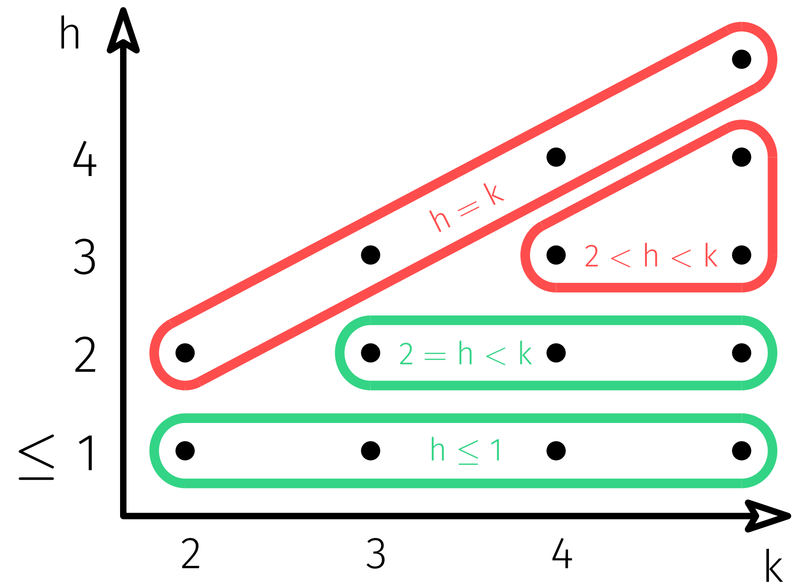

$k > 3$: Brute-force $k - 3$ quantifiers

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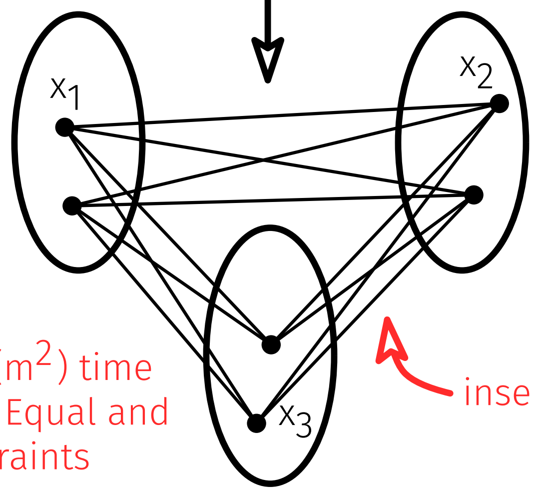

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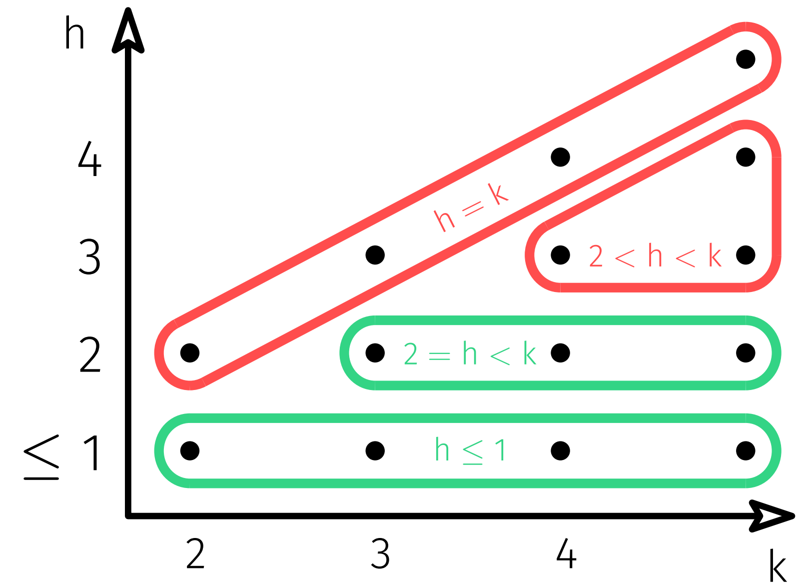

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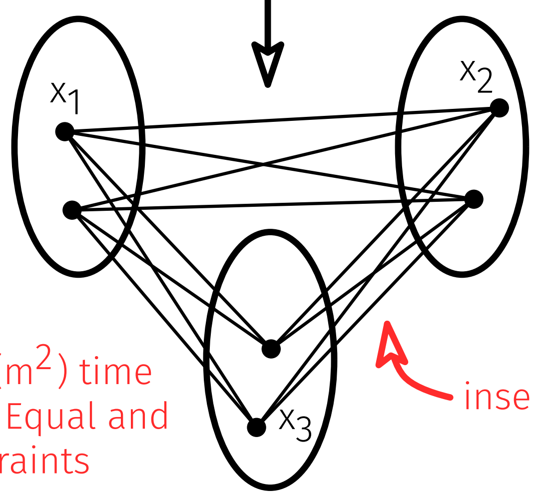

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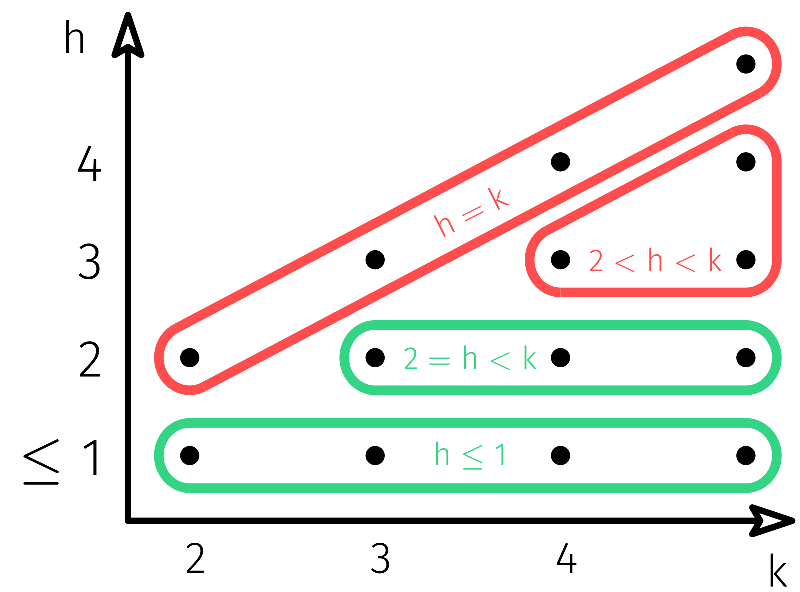

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Idea: repeat the above reduction and combine the triangle constraints

Algorithms for Properties of Hardness $h \leq 2$

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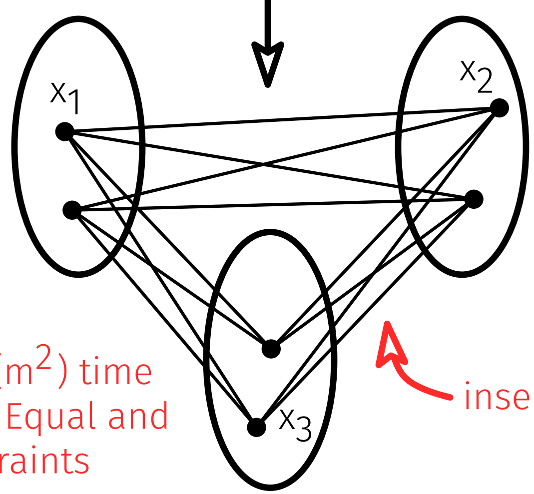

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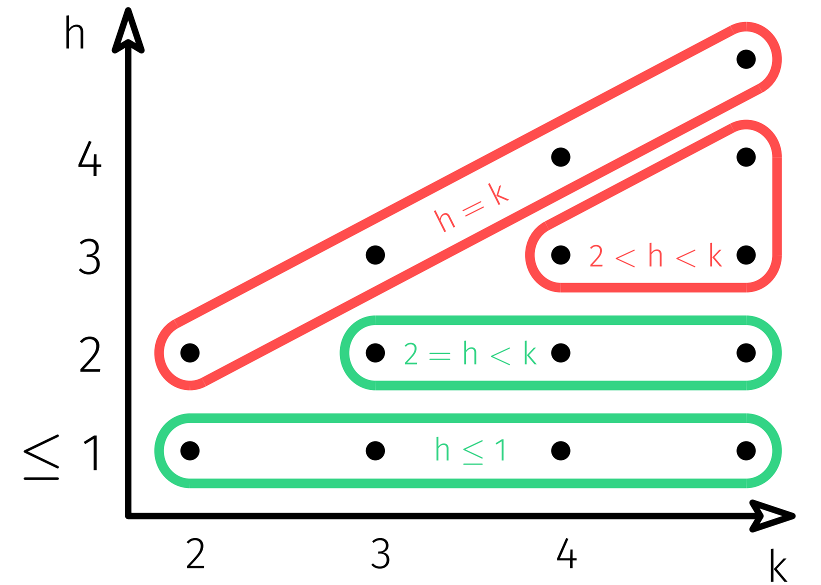

Examples

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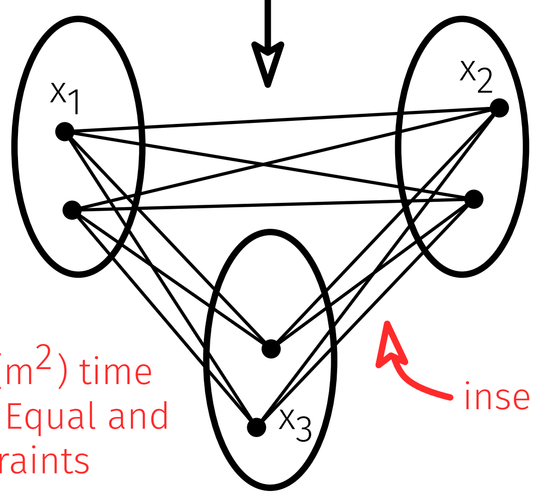

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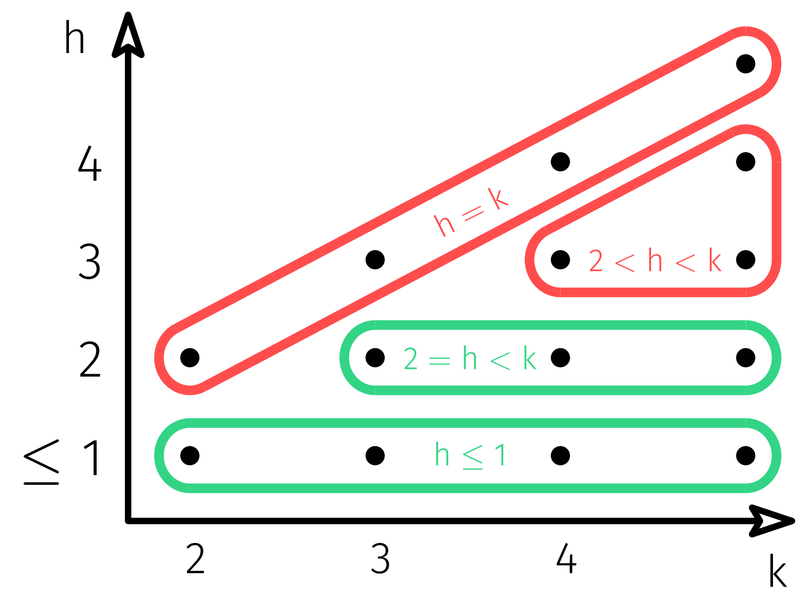
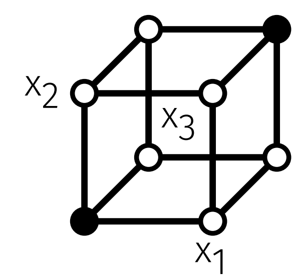


Examples

$$\exists x_1 \; \exists x_2 \; \exists x_3 \; \forall i: \quad \mathrm{NAE}(x_1[i], x_2[i], x_3[i])$$



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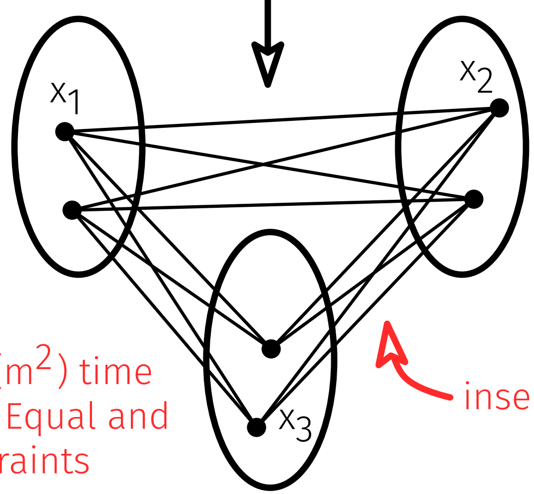

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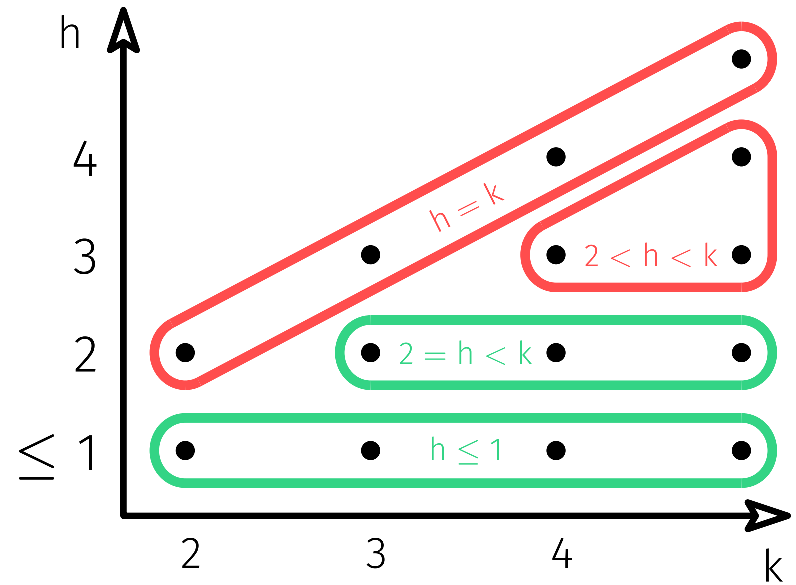

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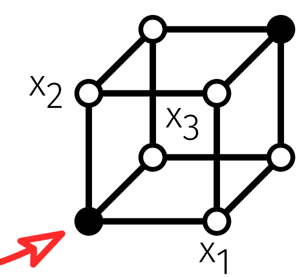
Examples

$$\exists x_1\ \exists x_2\ \exists x_3\ \forall i: \mathrm{NAE}(x_1[i], x_2[i], x_3[i])$$

falsifying assignments



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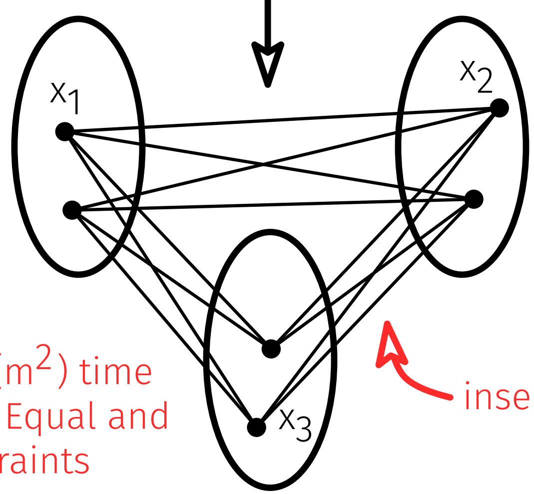

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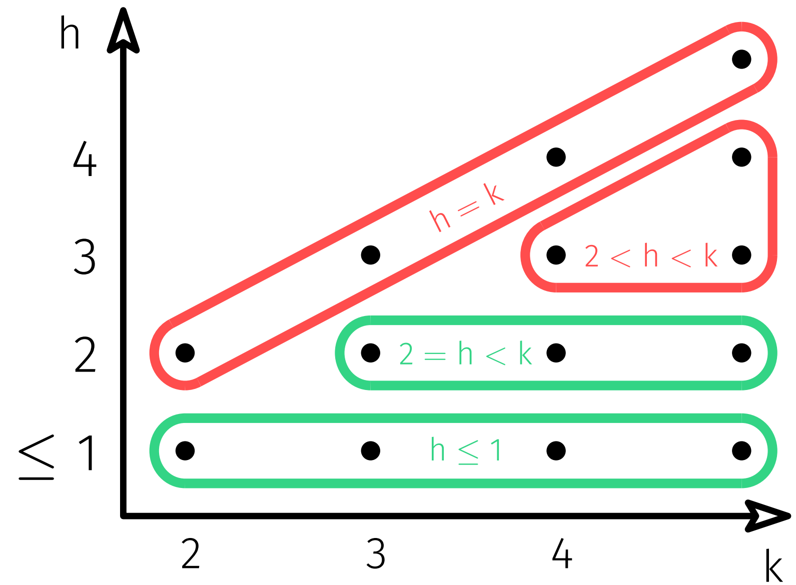

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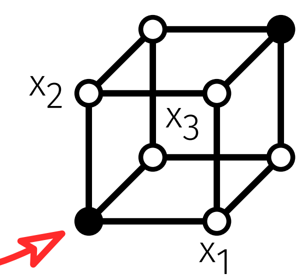
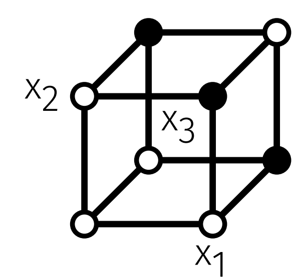
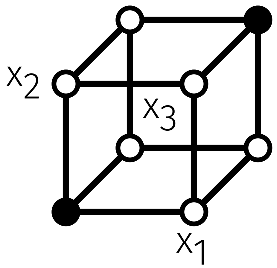


falsifying assignments

$$\exists x_1\ \exists x_2\ \exists x_3\ \forall i: x_1[i] + x_2[i] + x_3[i] \neq 2$$



How to Employ Sum and Equal Constraints



$$\exists x_1 \;\; \exists x_2 \;\; \exists x_3 \;\; \forall i:$$
$$\mathrm{NAE}(x_1[i], x_2[i], x_3[i])$$

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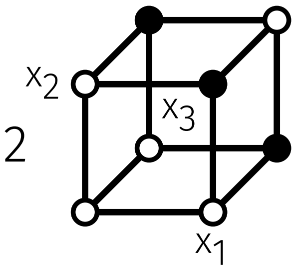

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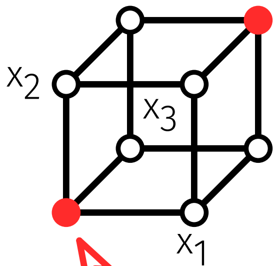

$$\exists x_1 \ \exists x_2 \ \exists x_3 \ \forall i:$$
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Idea: Exclude 000 and 111 by a Sum constraint

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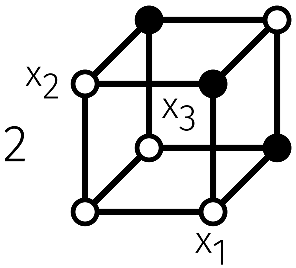

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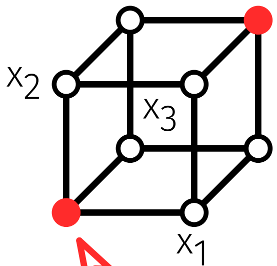

$\exists x_1\ \exists x_2\ \exists x_3\ \forall i:$
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Idea: Exclude 000 and 111 by a Sum constraint

Exclude 111 and 000 in all dimensions

$\Leftrightarrow\ 0 = \|x_1 \wedge x_2 \wedge x_3\|_1 = \|\bar{x}_1 \wedge \bar{x}_2 \wedge \bar{x}_3\|_1$

$\exists x_1\ \exists x_2\ \exists x_3\ \forall i:$
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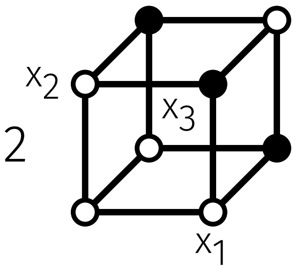

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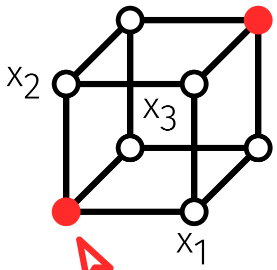

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$\Leftrightarrow\ 0 = \|x_1 \wedge x_2 \wedge x_3\|_1 + \|\bar{x}_1 \wedge \bar{x}_2 \wedge \bar{x}_3\|_1$

$\exists x_1\ \exists x_2\ \exists x_3\ \forall i:$
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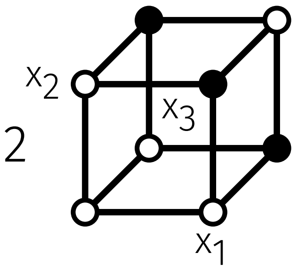
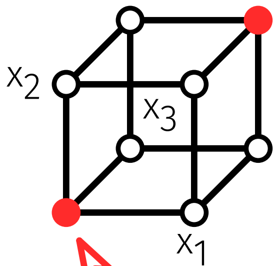


How to Employ Sum and Equal Constraints



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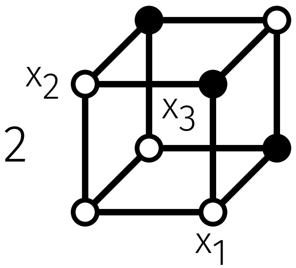
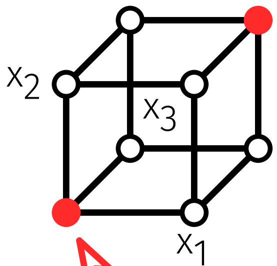
$$\Leftrightarrow\ 0 = \|x_1 \wedge x_2 \wedge x_3\|_1 + \|\bar{x}_1 \wedge \bar{x}_2 \wedge \bar{x}_3\|_1$$

$$\begin{aligned}\Leftrightarrow\ 0 &= \|x_1 \wedge x_2 \wedge x_3\|_1 - \|x_1 \wedge x_2 \wedge x_3\|_1 \\ &+ \|x_1 \wedge x_2\|_1 - \|x_1\|_1 \\ &+ \|x_2 \wedge x_3\|_1 - \|x_2\|_1 \\ &+ \|x_3 \wedge x_1\|_1 - \|x_3\|_1 \\ &+ d\end{aligned}$$

$$\exists x_1\ \exists x_2\ \exists x_3\ \forall i:$$
$$x_1[i] + x_2[i] + x_3[i] \neq 2$$



How to Employ Sum and Equal Constraints



$$\exists x_1\ \exists x_2\ \exists x_3\ \forall i:$$
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Idea: Exclude 000 and 111 by a Sum constraint

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cancels!

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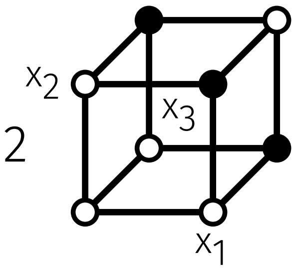

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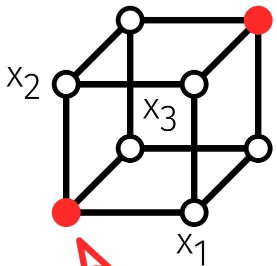

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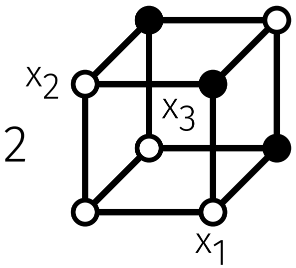

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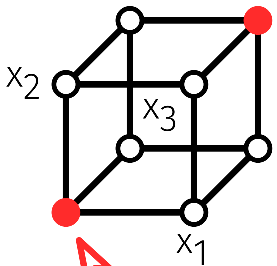
$$\Leftrightarrow \quad 0 = \|x_1 \wedge x_2 \wedge x_3\|_1 + \|\bar{x}_1 \wedge \bar{x}_2 \wedge \bar{x}_3\|_1$$

$$\begin{aligned}
\Leftrightarrow \quad 0 = &\boxed{\|x_1 \wedge x_2 \wedge x_3\|_1 - \|x_1 \wedge x_2 \wedge x_3\|_1} \quad \text{cancels!} \\
&+ \boxed{\|x_1 \wedge x_2\|_1 - \|x_1\|_1} \; w(x_1, x_2) \\
&+ \boxed{\|x_2 \wedge x_3\|_1 - \|x_2\|_1} \; w(x_2, x_3) \\
&+ \boxed{\|x_3 \wedge x_1\|_1 - \|x_3\|_1} \; w(x_3, x_1) \\
&+ \boxed{d} \; \text{target } t
\end{aligned}$$

How to Employ Sum and Equal Constraints



$\exists x_1 \ \exists x_2 \ \exists x_3 \ \forall i$:
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Idea: Exclude 000
and 111 by a Sum
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$$\Leftrightarrow \quad 0 = \boxed{\|x_1 \wedge x_2 \wedge x_3\|_1 - \|x_1 \wedge x_2 \wedge x_3\|_1}$$

cancels!

$$+ \boxed{\|x_1 \wedge x_2\|_1 - \|x_1\|_1} \ w(x_1, x_2)$$

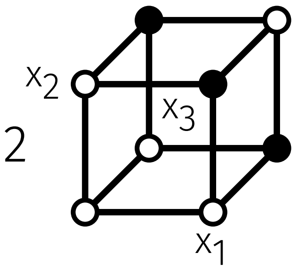
$$+ \boxed{\|x_2 \wedge x_3\|_1 - \|x_2\|_1} \ w(x_2, x_3)$$

$$+ \boxed{\|x_3 \wedge x_1\|_1 - \|x_3\|_1} \ w(x_3, x_1)$$

$$+ \boxed{d} \ \text{target } t$$

Generalizes for any pair of falsifying assignments of odd Hamming distance

$\exists x_1 \ \exists x_2 \ \exists x_3 \ \forall i$:
$x_1[i] + x_2[i] + x_3[i] \neq 2$



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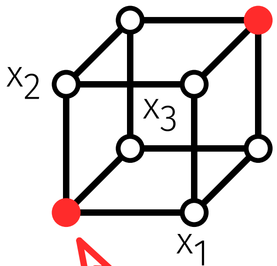

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Idea: Exclude 000
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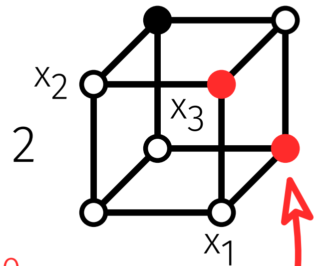

Idea: Exclude 110
and 101 by an Equal
constraint

Exclude 111 and 000 in all dimensions

$$\Leftrightarrow\ 0 = \|x_1 \wedge x_2 \wedge x_3\|_1 = \|\bar{x}_1 \wedge \bar{x}_2 \wedge \bar{x}_3\|_1$$

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cancels!

$$+ \boxed{\|x_1 \wedge x_2\|_1 - \|x_1\|_1}\ w(x_1, x_2)$$

$$+ \boxed{\|x_2 \wedge x_3\|_1 - \|x_2\|_1}\ w(x_2, x_3)$$

$$+ \boxed{\|x_3 \wedge x_1\|_1 - \|x_3\|_1}\ w(x_3, x_1)$$

$$+ \boxed{d}\ \text{target } t$$

Generalizes for any pair of falsifying
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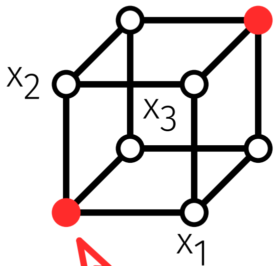
How to Employ Sum and Equal Constraints



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Idea: Exclude 000 and 111 by a Sum constraint

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$$\Leftrightarrow \quad 0 = \underbrace{\boxed{\|x_1 \wedge x_2 \wedge x_3\|_1 - \|x_1 \wedge x_2 \wedge x_3\|_1}}_{\text{cancels!}}$$
$$+ \boxed{\|x_1 \wedge x_2\|_1 - \|x_1\|_1} \ w(x_1, x_2)$$
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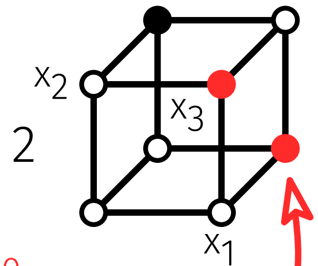
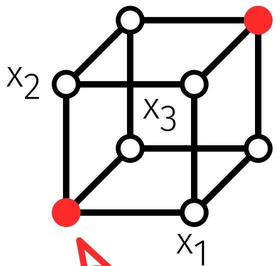


Idea: Exclude 110 and 101 by an Equal constraint

Exclude 110 and 101 in all dimensions

$$\Leftrightarrow \quad x_1 \wedge x_2 \ = \ x_1 \wedge x_3$$

How to Employ Sum and Equal Constraints



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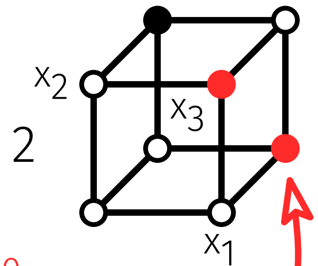

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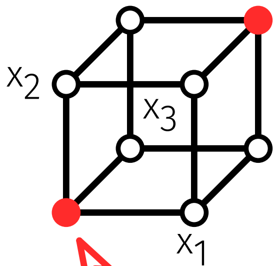

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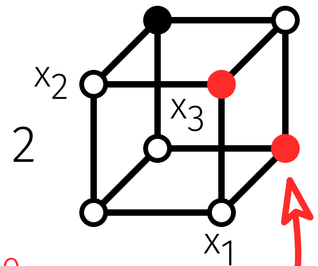

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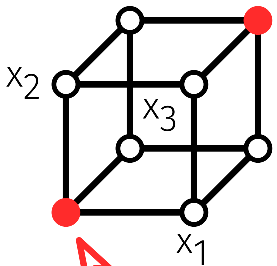
convert vectors arbitrarily into numbers

How to Employ Sum and Equal Constraints



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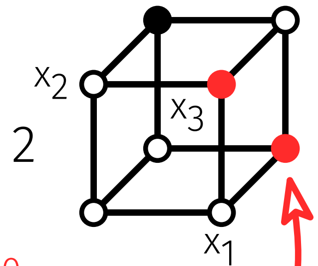

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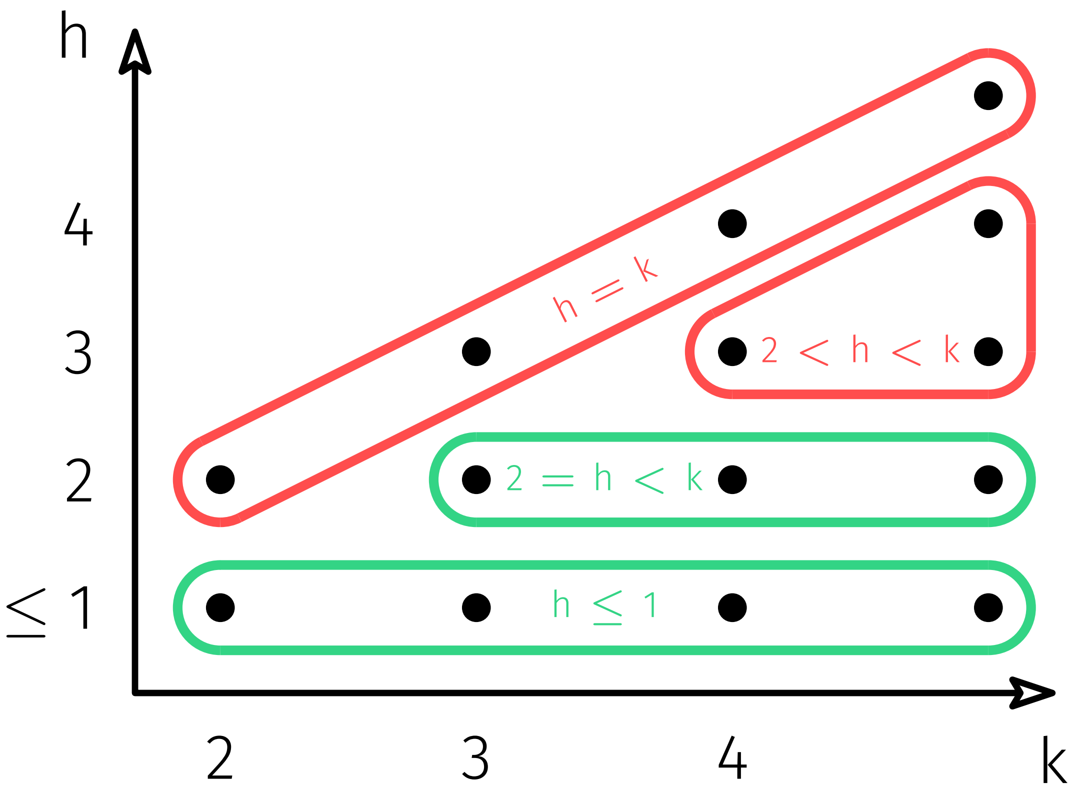
Conclusion and Open Problems



Open problems

- In which way does the classification extend to first-order queries beyond $\exists^k\forall$-quantified graph properties?
- What's the exact complexity of low-hardness properties?
- Equivalence of finding cliques in h-hypergraphs and properties of hardness h?

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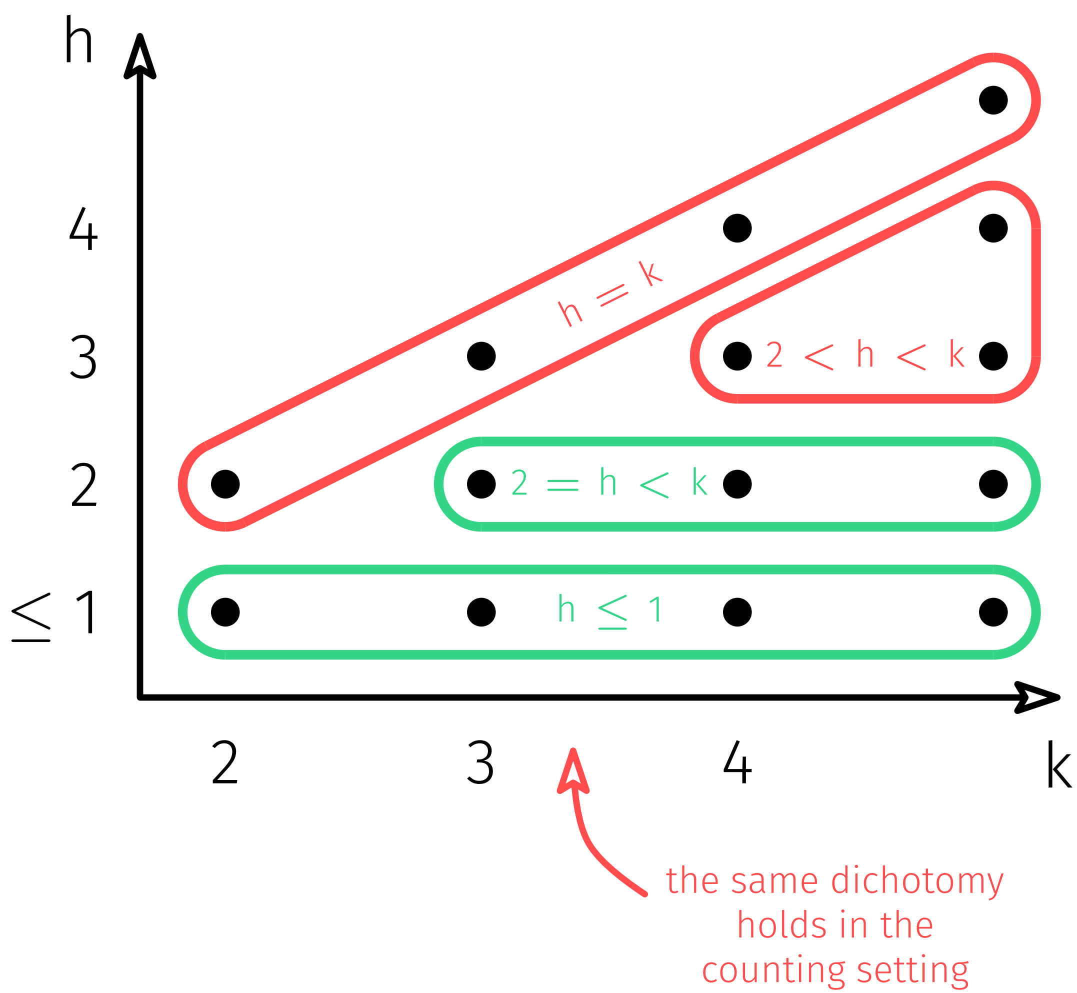

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