Near-Optimal Pseudorandom Generators for Constant-Depth Read-Once Formulas

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[2] Supported by a Simons Investigator Award (#409864, David Zuckerman)

[3] Supported by the NSF GRFP under Grant DGE-1610403 and by a Harrington Fellowship from UT Austin

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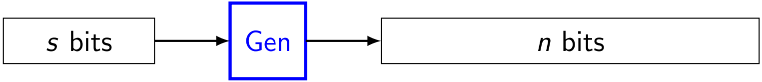

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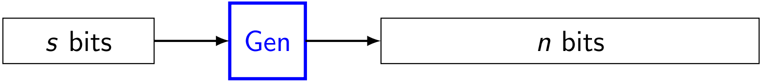

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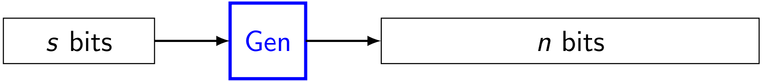

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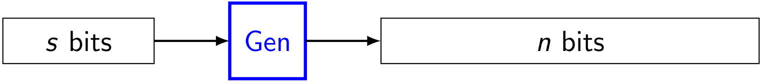

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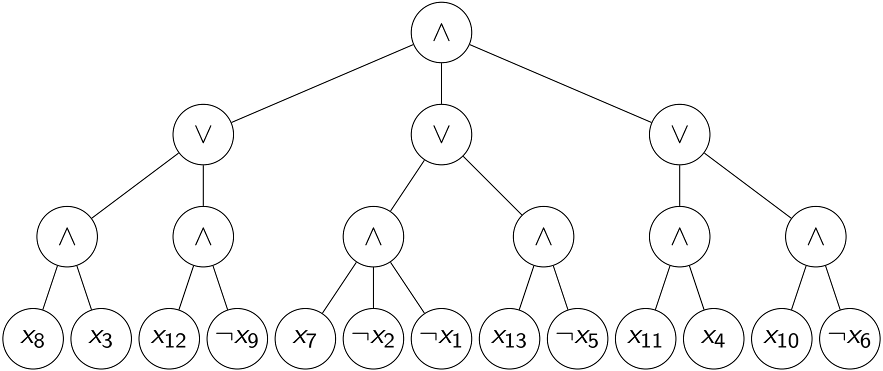
- Minimize seed length $s = s(n, \varepsilon)$

Read-once formulas



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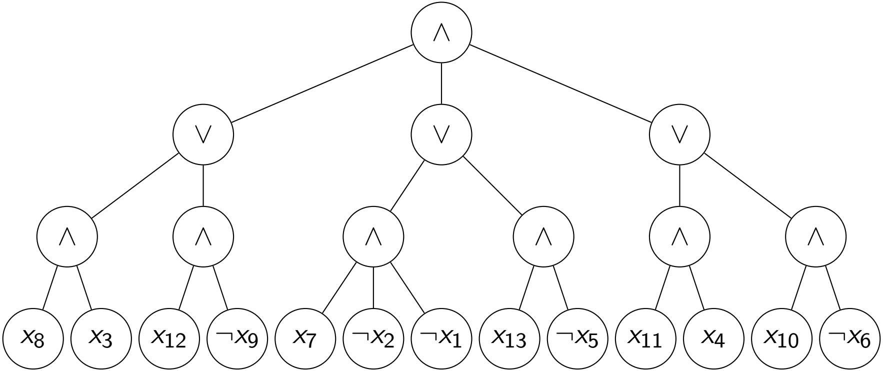

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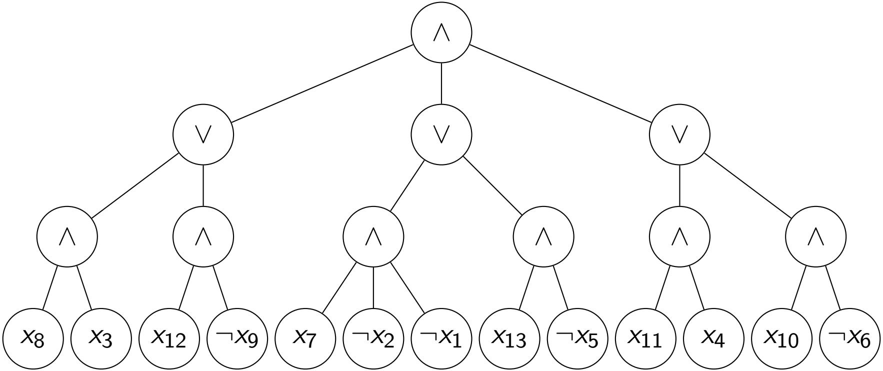

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- Read-once version of $\mathbf{AC}^0$

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- **Main result:** PRG for read-once $\mathbf{AC}^0$ with seed length

$$\log(n/\varepsilon) \cdot O(d \log\log(n/\varepsilon))^{2d+2}.$$

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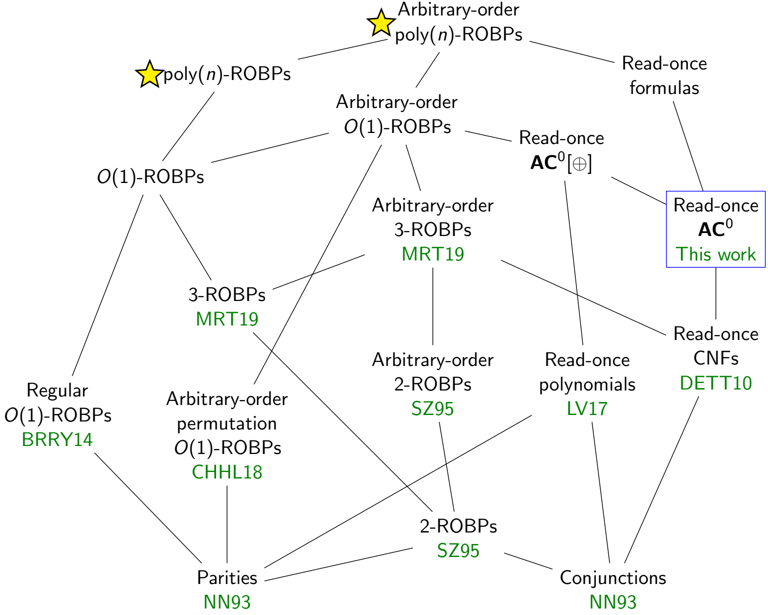
- Read-once $\mathbf{AC}^0$ is one of the frontiers of this progress

Seed length $\widetilde{O}(\log n)$



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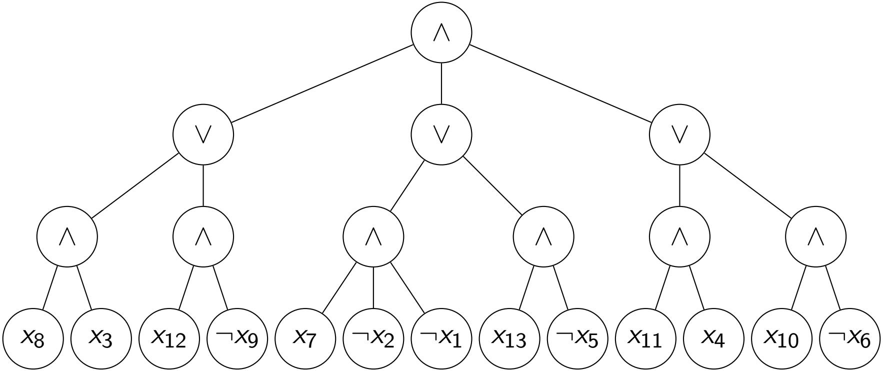

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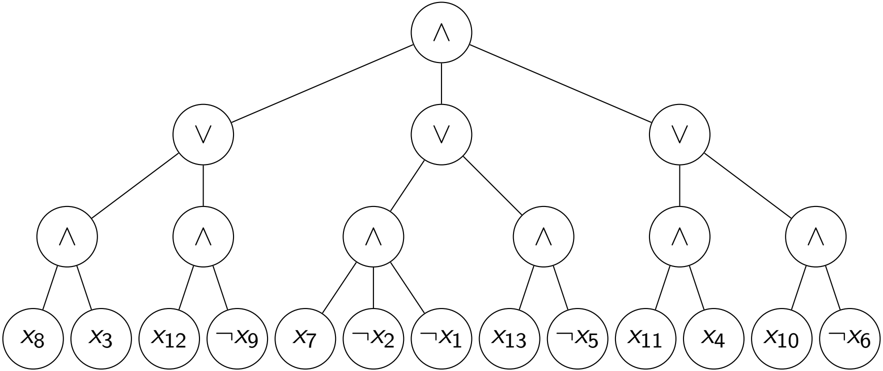

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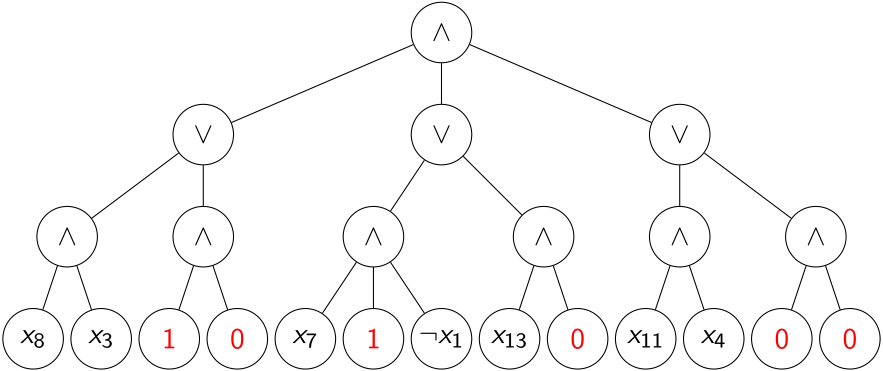

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- Define $\mathsf{Res}\colon \{0,1\}^n \times \{0,1\}^n \to \{0,1,\star\}^n$ by

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$y =$	0	1	1	0	0	1	0	0
$z =$	0	0	1	1	1	1	0	1
$\mathrm{Res}(y,z) =$	0	$\star$	$\star$	1	1	$\star$	0	1

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- (Proof involves clever Fourier analysis, building on [RSV13, HLV18, CHRT18])

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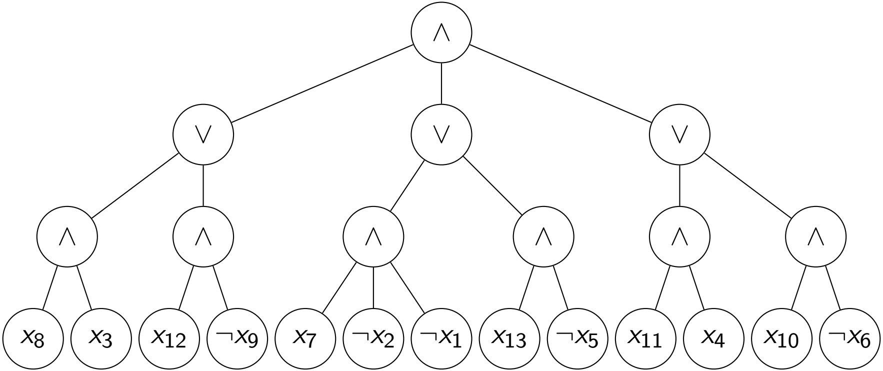

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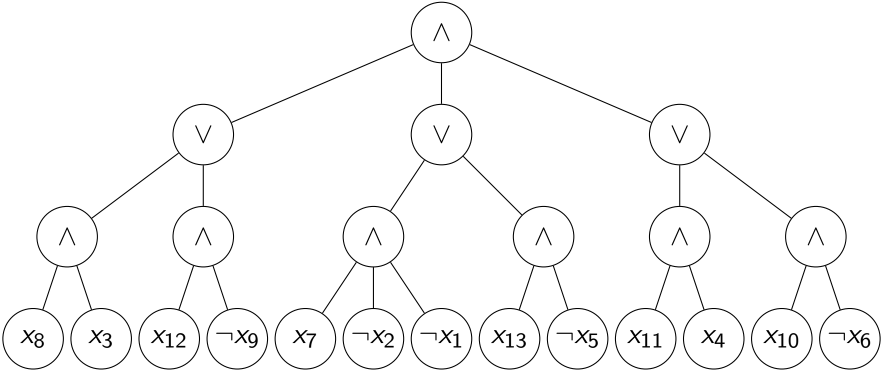

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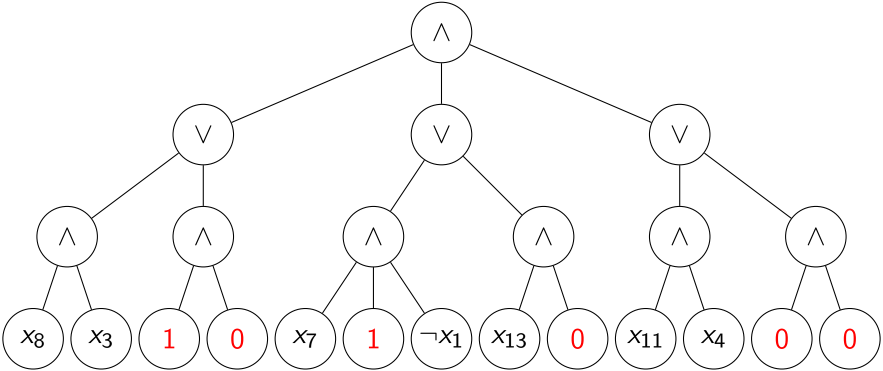

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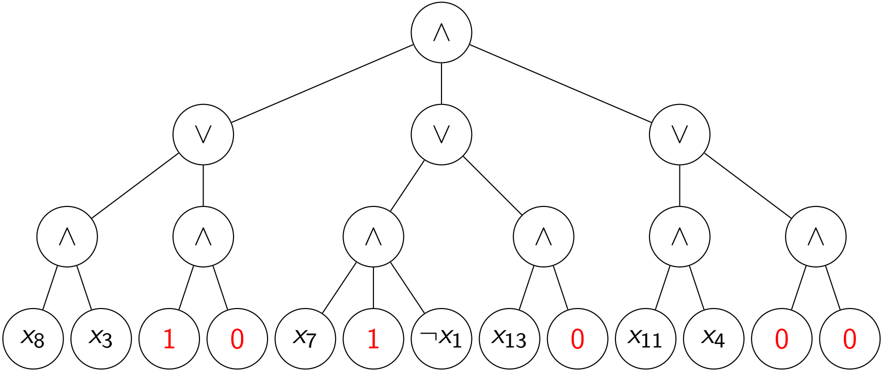

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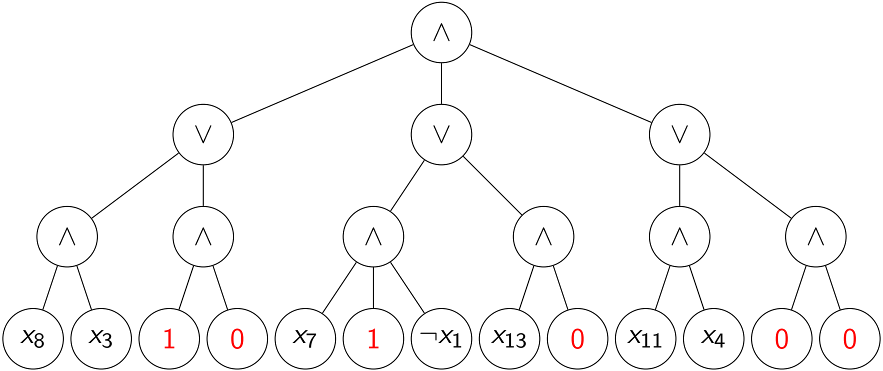

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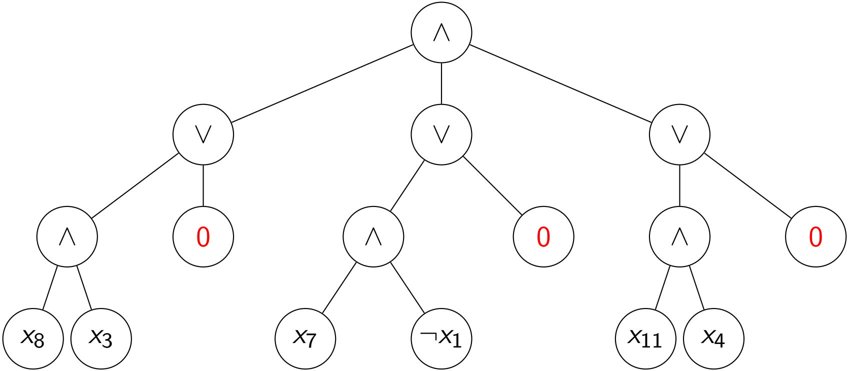

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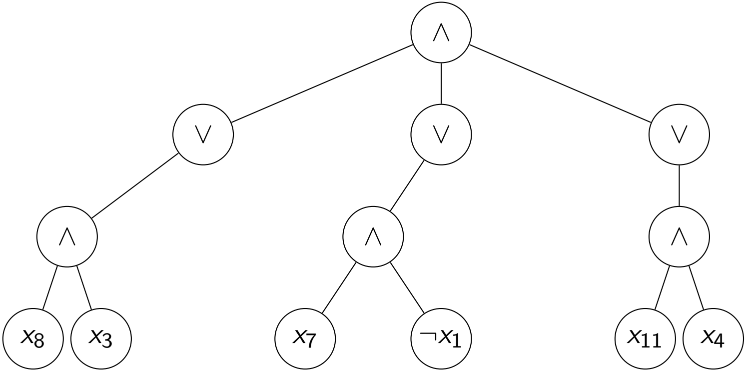

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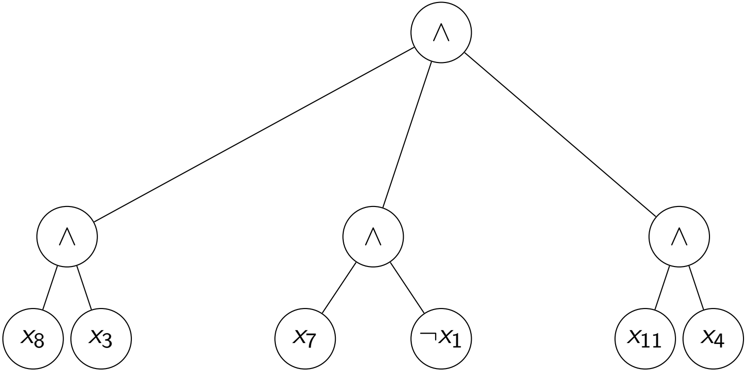

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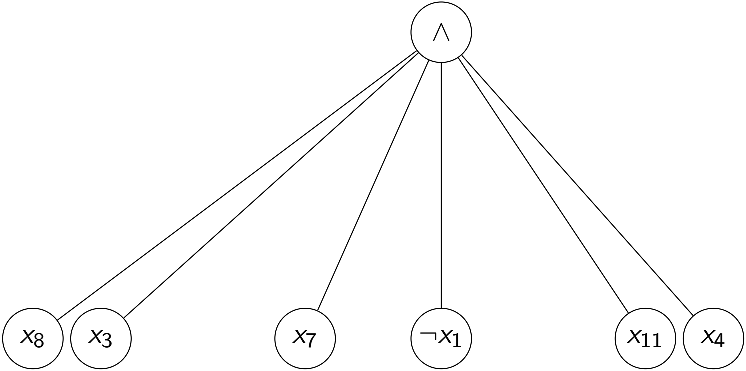

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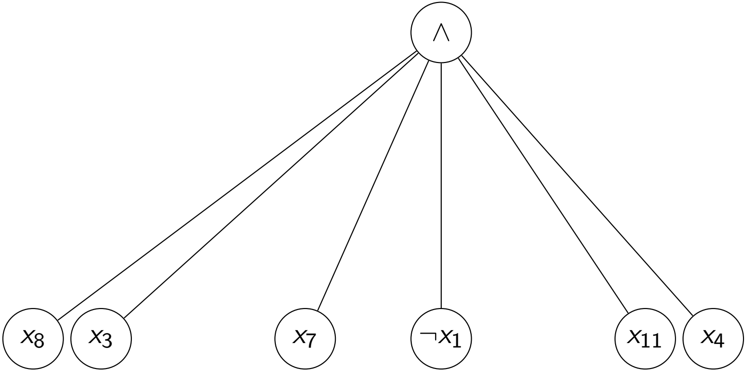

- Step 2: Fool restricted formula, taking advantage of simplicity

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3. $X = \mathrm{Res}(G_d \oplus D, G'_d \oplus D')$

Preserving expectation

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- So we can simply apply Forbes-Kelley result:

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Simplification

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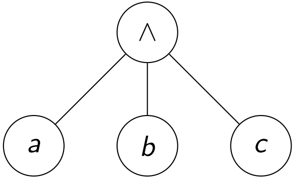
- **Lemma**: Can be decided in depth-d read-once $\mathbf{AC}^0$

Deciding whether $f|_{\text{Res}(y,z)} \equiv b$



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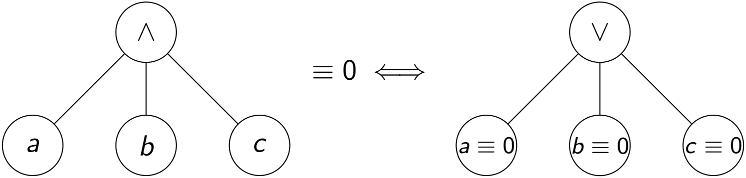

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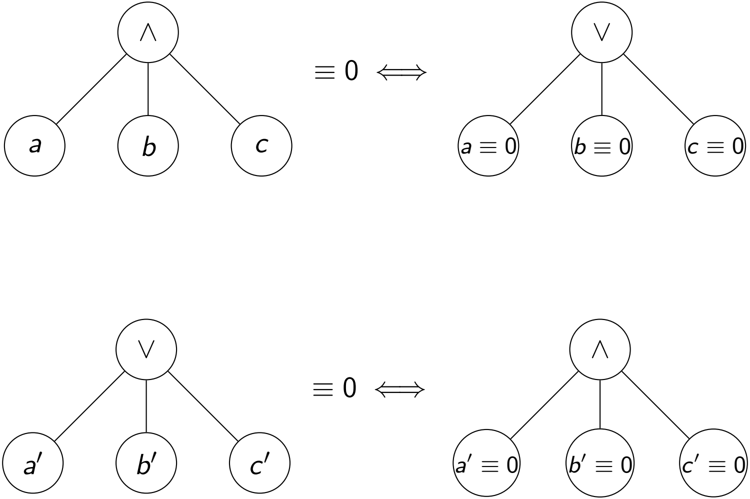

Deciding whether $f|_{\mathrm{Res}(y,z)} \equiv b$ (continued)

- At bottom, we get one additional layer:

$$(\mathrm{Res}(y,z)_i \equiv b) \iff (y_i = 0 \wedge z_i = b)$$
$$(\neg\,\mathrm{Res}(y,z)_i \equiv b) \iff (y_i = 0 \wedge z_i = 1 - b)$$

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- Hybrid argument:

$$\Pr_{X^{\circ t}}[f|_{X^{\circ t}} \equiv b] \approx \Pr_{R^{\circ t}}[f|_{R^{\circ t}} \equiv b]$$

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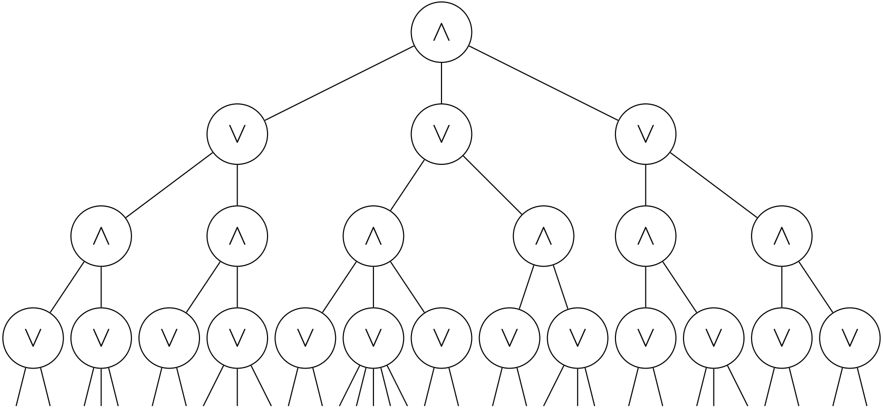
Idea of proof that $\Delta \mapsto \text{polylog}\, n$

Total depth $d + 1$



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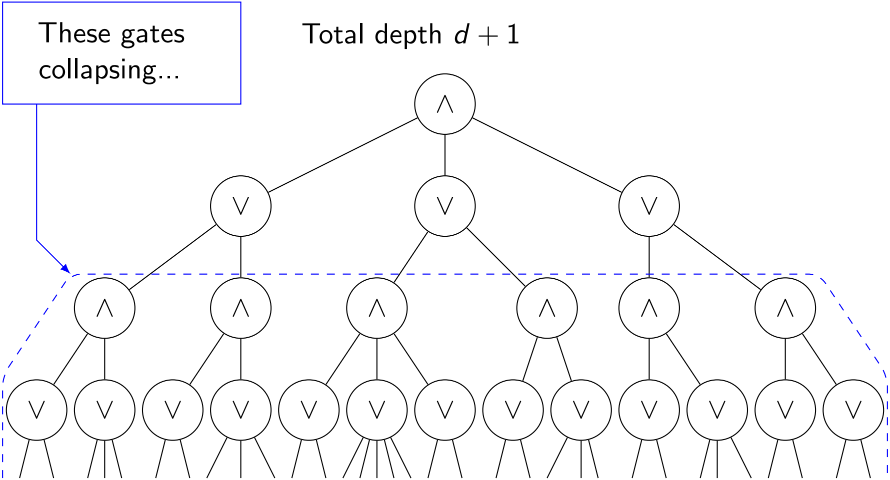

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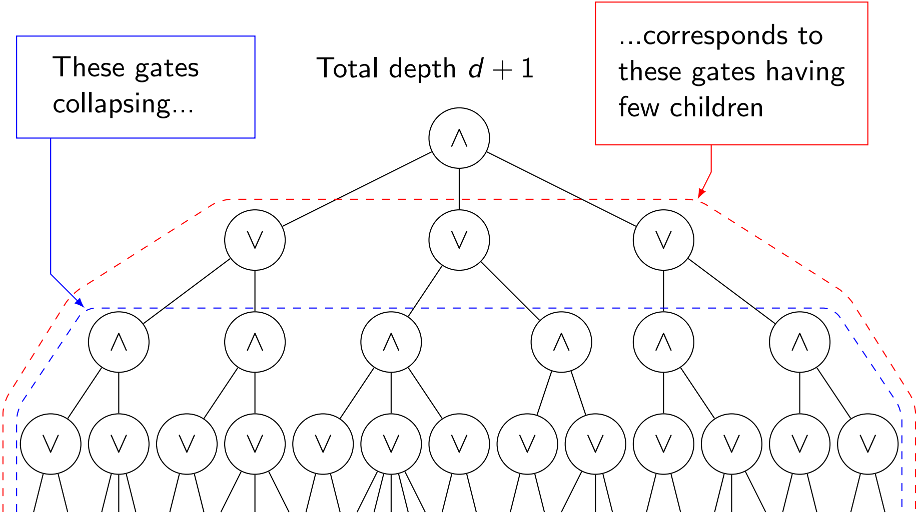

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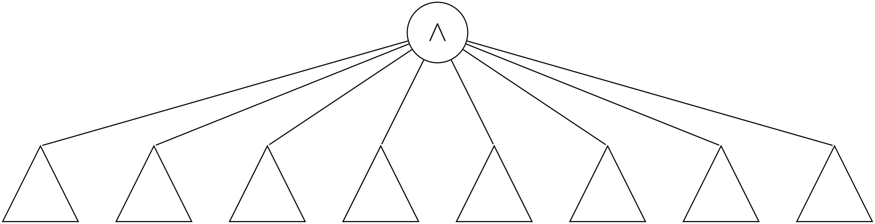

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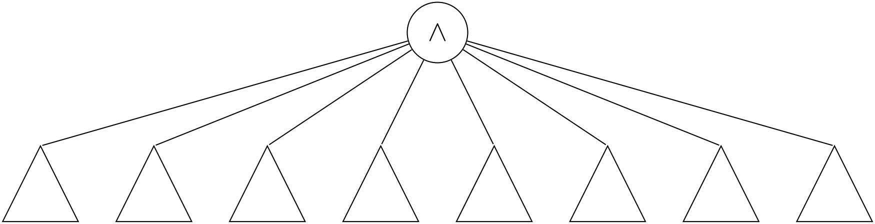
- To recap, after $t = O((\log\log n)^2)$ restrictions, $\Delta = \text{polylog } n$
- Total cost so far: $\widetilde{O}(\log n)$ truly random bits



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Directions for further research

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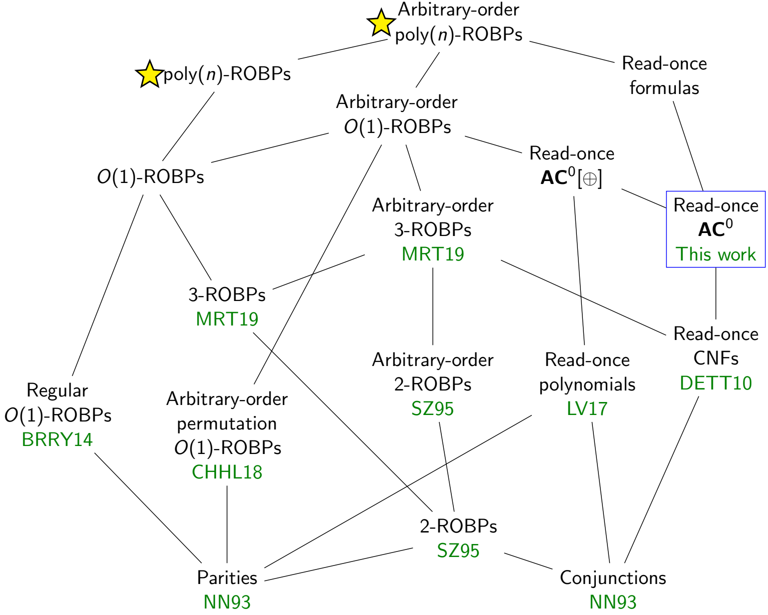

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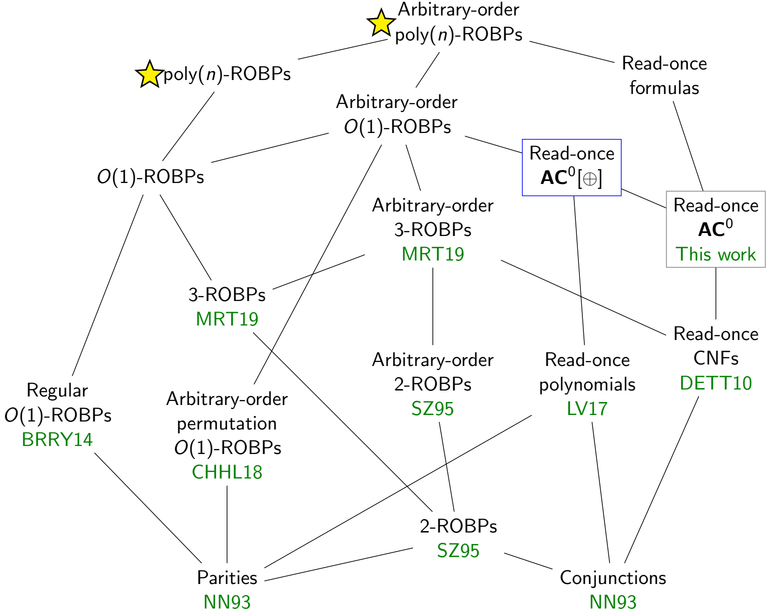
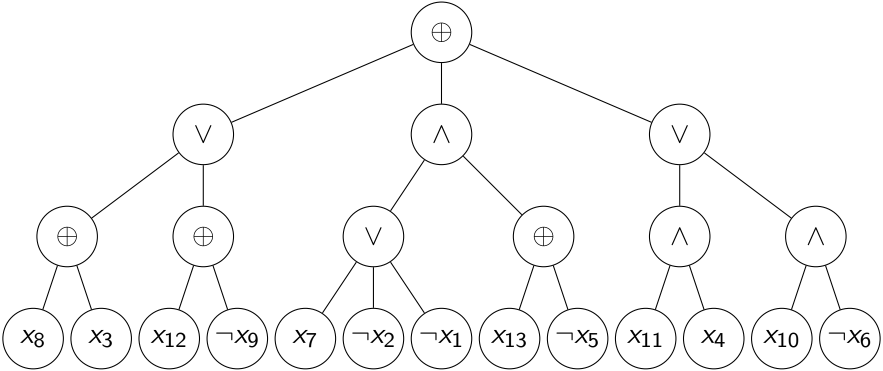


Read-once $\mathbf{AC}^0[\oplus]$



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- Thanks! Questions?