The following set of problems is a supplement to the series of lectures on Operator Scaling given by Avi Wigderson at CCC 2017, and were compiled with the help of Ankit Garg and Rafael Oliveira. They are not meant in any way to exhaust all aspects covered in these lectures, but rather to exemplify the diverse set of questions which motivated and guided us during the course of this research. This set of problems is divided into four sections, each highlighting a different area of research related to operator scaling.

The questions are of varied difficulty, but hopefully self contained, and are not ordered by level of difficulty. We provided hints and/or references to most of the questions, so the reader can easily find the answer to all questions in this problem set.

1 Non-commutative algebra

Definition 1.1 (Hall Blocker)**.** Let R be a ring (not necessarily commutative), $M_n(R)$ be the space of $n \times n$ matrices with entries in R and $r \in \mathbb{N}$ be such that $0 \le r < n$. We say that a matrix $A \in M_n(R)$ has an *r-Hall blocker* (or simply a Hall blocker) if A has a zero submatrix of dimensions $i \times j$ s.t. $i + j \ge 2n - r$.

The weak algorithm (e.g. Chapter 2 in [Coh06]) is a generalization of the division algorithm for univariate polynomials to multivariate non-commutative polynomials. No such analogue exists for multivariate commutative polynomials.

Problem 1.2 (**Weak algorithm**)**.** *Let $p, q_1, \ldots, q_n \in \mathbb{F}\langle \mathbf{x} \rangle$ be polynomials in the non-commutative variables $\mathbf{x} = (x_1, \ldots, x_m)$. p is said to be right degree-dependent on $q_1, \ldots, q_n$ if $p = 0$ or there exist polynomials $r_1, \ldots, r_n$ s.t.*

$$\deg\left(p - \sum_{i=1}^{n} q_i r_i\right) < \deg(p)$$

and

$$\deg(q_i) + \deg(r_i) \le \deg(p)$$

for all i. Polynomials $q_1, \ldots, q_n$ are said to be right degree-dependent if there exist polynomials $r_1, \ldots, r_n$ s.t.

$$\deg\left(\sum_{i=1}^{n} q_i r_i\right) < \max_i \{\deg(q_i) + \deg(r_i)\}$$

Prove that if $q_1, \ldots, q_n$ are right degree-dependent, then there exists one of them which is right degree-dependent on rest of them.

Next problem is about the gap between commutative and non-commutative ranks. The result is due to [FR04] following a lemma in [Fla62].

Problem 1.3 (**Commutative vs Non-commutative rank - upper bound**)**.** *Let $L = \sum_{i=1}^{m} x_i A_i$ be an $n \times n$ symbolic matrix where the variables x_i could be commuting or non-commuting. Let* $\text{rank}(L)$ *denote its rank over the field of commutative rational functions $\mathbb{F}(\mathbf{x})$ and let* $\text{nc-rank}(L)$ *denote its rank over the free skew field $\mathbb{F} \langle\!(\mathbf{x})\!\rangle$. Prove that* $\text{nc-rank}(L) \le 2 \cdot \text{rank}(L)$.

Hint: Use the following theorem due to Cohn [Coh95] and Fortin-Reutenauer [FR04]: $\text{nc-rank}(L) <= r$ iff there exist invertible matrices $B, C \in M_n(\mathbb{F})$ s.t. BLC contains an r-Hall blocker.

Problem 1.4 (**Commutative vs Non-commutative rank - lower bound**). *The* 3×3 *(generic) skew-symmetric matrix gives a gap of* $3/2$*. Construct an example with gap* $> 3/2$*.*

Recently, [DM16], following [LO15], constructed examples with gap approaching arbitrarily close to 2.

The next problem is about a non-commutative analogue of Valiant's completeness of determinant for formulas. This was proved in [HW14]. A similar statement also appears in Malcolmson's approach to construction of the free skew field [Mal78].

Problem 1.5. *Let* $f \in \mathbb{F}\langle\!\langle \mathbf{x} \rangle\!\rangle$ *be a rational function in non-commuting variables computable by a non-commutative formula* Φ *(with divisions) of size* s*. Then there exists an* $s' \times s'$ *(*$s' \leq 2s$*) matrix* A_Φ*, invertible over* $\mathbb{F}\langle\!\langle \mathbf{x} \rangle\!\rangle$*, whose entries are either variables or elements of* $\mathbb{F}$ *s.t.* f *is the* $(1,1)$ *entry of* A_Φ^{-1}*.*

Problem 1.6 (**Hua's identity**). *Prove that for non-commuting variables* x, y*,*

$$\left(x + xy^{-1}x\right)^{-1} + (x+y)^{-1} = x^{-1}$$

Problem 1.7 (**Higman's trick**). *Given a matrix* $M \in M_n(\mathbb{F}[\mathbf{x}])$ *(or* $\in M_n(\mathbb{F}\langle \mathbf{x} \rangle)$*), find a map that linearizes the matrix* M*, i.e. the entries of the new matrix should be affine forms, and preserves its* co-rank *(or* co-nc-rank*). If the entries of* M *have small formula size, then the map shouldn't blow up the dimension of* M *by a lot.*

Problem 1.8. *One of Cohn's theorems* [Coh95] *says that for a linear matrix (whose entries are affine forms)* $L \in M_n(\mathbb{F}\langle \mathbf{x} \rangle)$*,* $\text{nc-rank}(L) \leq r$ *iff there exist* $n \times r$ *and* $r \times n$ *linear matrices s.t.* $L = L_1 L_2$*. Use this theorem to give an exponential time algorithm for computing* $\text{nc-rank}(L)$*.*

Problem 1.9 (**Proof of Cohn's theorem**). *In this exercise, we will prove Cohn's theorem*[1] *that* $\text{nc-rank}(L) \leq r$ *iff there exist invertible matrices* $B, C \in M_n(\mathbb{F})$ *s.t.* BLC *contains an* r*-Hall blocker. Our starting point will the non-trivial theorem*[2] *that* $\text{nc-rank}(L) \leq r$ *iff there exist matrices* $K_1 \in M_{n\times r}(\mathbb{F}\langle \mathbf{x} \rangle)$ *and* $K_2 \in M_{r \times n}(\mathbb{F}\langle \mathbf{x} \rangle)$ *(not necessarily linear) s.t.* $L = K_1 K_2$*.*

1. *Suppose* $p_1, \ldots, p_n$ *and* $q_1, \ldots, q_n$ *are non-commutative polynomials such that*

$$\sum_{i=1}^{n} p_i q_i = f \tag{1}$$

Then there exists a matrix $E \in GL_n(\mathbb{F}\langle \mathbf{x} \rangle)$ *(the entries of* E^{-1} *will also have polynomial entries) which "kills all cancellations" in equation (1). Formally, if*

$$\begin{bmatrix} p_1' & p_2' & \ldots & p_n' \end{bmatrix} = \begin{bmatrix} p_1 & p_2 & \ldots & p_n \end{bmatrix} E$$

and

$$\begin{bmatrix} q_1' \\ q_2' \\ \vdots \\ q_n' \end{bmatrix} = E^{-1} \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{bmatrix}$$

[1] Technically Cohn only proved it for $r = n-1$ and the full theorem appeared in *[FR04]*

[2] Again due to Cohn

then

$$\sum_{i=1}^{n} p_i' q_i' = f$$

and for each i, either one of p_i' or q_i' is 0, or $deg(p_i') + deg(q_i') \leq deg(f)$. (Hint*: use the weak algorithm*).

2. *If there exist K_1, K_2 as above s.t. $L = K_1 K_2$, then prove that there exist linear L_1, L_2 with the same dimensions s.t. $L = L_1 L_2$. Furthermore, for every $1 \leq i \leq r$, either the i^{th} column of L_1 has only elements from $\mathbb{F}$ or the i^{th} row of L_2 does. Note that this does not mean that $L_1 \in M_{n \times r}(\mathbb{F})$ (think of a counterexample).* (Hint*: Reduce to the above lemma by introducing auxilliary non-commuting variables*)

3. *Use the above to construct a Hall blocker for L.*

The following connection between non-commutative rank and tensor rank is due to [DM16].

Problem 1.10 (**Non-commutative rank and tensor rank**)**.** *Let the linear matrix $L = \sum_{i=1}^{n} x_i A_i \in M_n(\mathbb{F}\langle \mathbf{x} \rangle)$ be s.t.* $\text{rank}(L) = \alpha n$ *and* $\text{nc-rank}(L) = n$*. Suppose $B_1, \ldots, B_n$ be matrices in $M_n(\mathbb{F})$ s.t. $\sum_{i=1}^{n} A_i \otimes B_i$ is full rank (they exist due to the recent bounds of* [DM15, IQS15]*). Let B be the $n \times n \times n$ tensor whose slices are $B_1, \ldots, B_n$. Then prove that the tensor rank of B is at least n/α.*

The same bound also applies for border rank. With the examples in [DM16], this gives the best known lower bounds for border rank. These examples were discovered by Landsberg-Ottaviani [LO15] to prove lower bounds on border rank of the matrix multiplication tensor (they don't talk about non-commutative rank though and this connection was discovered by [DM16]).

2 Invariant theory

Problem 2.1 (**Wallace-Bolyai-Gerwien theorem**)**.** *Prove that a polygon in 2 dimensions can be transformed into another by cutting (along straight lines) into finite number of pieces and recomposing them by rotations and translations iff they have the same area.* (Hint*: Using triangulation, prove that a polygon of area A has in its orbit a rectangle of side lengths 1 and A.*)

Problem 2.2 (**Hilbert's third problem**)**.** *Hilbert conjectured that the same cut and paste strategy does not work for polyhedra in 3 dimensions of equal volume. This was proved by Max Dehn in 1900. Challenge yourself to prove that a regular tetrahedron cannot be transformed into a cube of equal volume.* (Hint*: construct an invariant other than volume using angles and edge-lengths!*)

Problem 2.3 (**Invariant rings**)**.** *Let $G \leqslant GL_m(\mathbb{F})$ be a linear group that acts on $\mathbb{F}^m$, which after fixing a basis for $\mathbb{F}^m$ can be thought of as acting on variables $\mathbf{x} = (x_1, \ldots, x_m)$. Prove that the set of invariant polynomials $\mathbb{F}[\mathbf{x}]^G$ is a subring of $\mathbb{F}[\mathbf{x}]$.*

Problem 2.4 (**Symmetric polynomials**)**.** *Hilbert in 1890 proved his famous theorem that invariant rings of linearly reductive group actions* (*includes all group actions one will ever encounter!*) *are finitely generated. Prove this theorem for the natural action of the symmetric group S_m on $\mathbb{F}^m$. In other words, prove that the ring of symmetric polynomials is generated by the m elementary symmetric ones.*

Problem 2.5. *What are the generators for the invariant ring of the following group action: $SL_n(\mathbb{C}) \times SL_n(\mathbb{C})$ acts on $M_n(\mathbb{C})$ by left-right multiplication i.e. $(A, B) : X \to AXB$.*

Let G act on a vector space V. Nullcone of the group action is defined as the set of common zeroes of all homogeneous invariant polynomials. Orbit of a vector $v \in V$ is the set $\{g \cdot v : g \in G\}$. Orbit-closure of v is the closure of the orbit in the Zariski topology. When the field is $\mathbb{C}$ and the group G is an algebraic group (like $GL_n(\mathbb{C})$), this coincides with closure in the Euclidean topology.

Problem 2.6. *What is the nullcone of the above left-right action? When are two matrices in the same orbit? When does one lie in the orbit-closure of the other? When do two orbit-closures intersect?*

Problem 2.7 (Orbits for finite groups). *Let G be a finite group that acts on a vector space V. Prove that two vectors $v_1, v_2 \in V$ are in the same orbit iff $p(v_1) = p(v_2)$ for all invariant polynomials $p \in \mathbb{F}[V]^G$.* (Hint*: Use Hilbert's Nullstellansatz.*)

Graph isomorphism is also an orbit problem and it is an excellent open question whether the above invariant theory approach can lead to efficient algorithms for it by giving a succinct description of the generating invariants.

The determinantal complexity of the $n \times n$ permanent is defined as the least m s.t. there is an affine map from the $n \times n$ matrix Y to an $m \times m$ matrix $A(Y)$ s.t. $\mathrm{per}(Y) = \det(A(Y))$. The least such m is denoted by $\mathrm{dc}(n)$. The VP vs VNP question is roughly equivalent to proving super-polynomial lower bounds on $\mathrm{dc}(n)$. The best known lower bound (over the complex numbers) is $\mathrm{dc}(n) \geq n^2/2$, due to [MR04].

If a group G acts on a vector space V, then this induces an action on polynomials $\mathbb{F}[V]$ as well as the degree d part $\mathbb{F}[V]_d$ via $g \cdot p(v) := p(g^{-1} \cdot v)$. The next problem considers such an action of $GL_m(\mathbb{C})$ on the space of polynomials over $M_m(\mathbb{C})$ induced by the action of $GL_m(\mathbb{C})$ on $M_m(\mathbb{C})$ by left multiplication. Let $X_{[n],[n]}$ denote top left $n \times n$ minor of X.

Problem 2.8 (Permanent vs Determinant). *Suppose the determinantal complexity of the $n \times n$ permanent is at most m. Then prove that the polynomial $X_{1,1}^{m-n}\mathrm{per}(X_{[n],[n]})$ lies in the orbit-closure of the polynomial* $\det(X)$.

Geometric Complexity Theory starts with the above expressing of the permanent vs determinant as an orbit-closure problem (the implication only goes one way) and aims to use tools from algebraic geometry and representation theory to resolve the orbit-closure problem.

3 Matrix scaling

Problem 3.1 (Hall's theorem). *Prove Hall's bipartite matching theorem. It says that an $n \times n$ bipartite graph has a perfect matching iff for every subset S of vertices on left side, $|N(S)| \geq |S|$, where $N(S)$ denotes the neighbours of S.*

Note that Hall's condition is equivalent to saying that the adjacency matrix A_G of a bipartite graph G has a Hall blocker iff G does not have a perfect matching. Observe the close connection to the hint in Problem 1.3. Another alternative formulation is that G has a perfect matching iff $\mathrm{per}(A_G) > 0$.

Problem 3.2 (**Doubly stochastic matrices**). *Prove that the polytope of doubly stochastic matrices is the convex hull of permutation matrices.*

Problem 3.3 (**Progress in matrix scaling**). *Let A be a non-negative matrix s.t. its row sums are 1 and $\mathrm{per}(A) > 0$. Suppose $\tilde{A}$ is obtained from A by scaling every column to make the column sums 1. Give a lower bound on $\mathrm{per}(\tilde{A})/\mathrm{per}(A)$ in terms of the l_2 distance of the column sums of A from the all 1's vector.* (Hint*: Eventually reduces to a quantitative version of the AM-GM inequality*)

The capacity of a non-negative matrix A is defined as follows [GY98]:

$$\mathrm{cap}(A) = \inf_{x>0:\prod_{i=1}^n x_i=1} \prod_{i=1}^{n} (Ax)_i$$

Problem 3.4. *Prove that $\mathrm{cap}(A) > 0$ iff the bipartite graph defined by support of A has a perfect matching.*

Problem 3.5 (**Capacity of doubly stochastic matrices**). *Prove that the capacity of a doubly stochastic matrix is 1.*

4 Operator scaling and Brascamp-Lieb inequalities

Given matrices $\mathbf{B} = (B_1, \ldots, B_m)$, where B_i is a $n_i \times n$ real matrix, the BL polytope $P_{\mathbf{B}}$ is the set of $\mathbf{p} = (p_1, \ldots, p_m)$ s.t. $p_i \geq 0$ and

1. $n = \sum_j p_j n_j$.
2. $\dim(V) \leq \sum_j p_j \dim(B_j(V))$ holds for all subspaces V of $\mathbb{R}^n$.

For a given BL datum $(\mathbf{B}, \mathbf{p})$, the BL constant $\mathrm{BL}(\mathbf{B}, \mathbf{p})$ is defined as the least constant C s.t. the following inequality works: for *every* tuple of m non-negative functions, $f = (f_1, f_2, \ldots, f_m)$, which are integrable according to the Lebesgue measure, we have the following inequality.

$$\int_{x\in\mathbb{R}^n} \prod_{j=1}^{m} (f_j(B_j x))^{p_j} dx \;\leq\; C \prod_{j=1}^{m} \left(\int_{x_j\in\mathbb{R}^{n_j}} f_j(x_j) dx_j \right)^{p_j}$$

Problem 4.1. *Express Hölder's inequality, Loomis-Whitney inequality and Shearer's lemma (in the continuous settings) as appropriate BL inequalities.*

Problem 4.2. *Prove that if $\mathbf{p} \notin P_{\mathbf{B}}$, then $\mathrm{BL}(\mathbf{B}, \mathbf{p}) = \infty$.*

The other direction is also true but non-trivial to prove.

Given a list of vectors $(v_1, \ldots, v_m)$ in $\mathbb{R}^n$, the linear matroid polytope corresponding to these vectors is the convex hull of indicator vectors of subsets $I \subseteq [m]$ of size n s.t. the corresponding vectors are linearly independent. Similarly given two list of vectors $(v_1, \ldots, v_m)$ and $(w_1, \ldots, w_m)$, the linear matroid intersection polytope corresponding to these is the convex hull of indicator vectors of subsets $I \subseteq [m]$ of size n s.t. the corresponding vectors are linearly independent in both lists.

Problem 4.3. *Write linear matroid and matroid intersection polytopes as special cases of* BL *polytopes. (you can use the fact that natural LP relaxations for these problems are integral.)*

Lieb proved that the optimizing functions in BL inequalities are Gaussian densities, wlog. This implies that

$$\mathrm{BL}(\mathbf{B},\mathbf{p}) = \left[\sup \frac{\prod_j (\det X_j)^{p_j}}{\det\left(\sum_j p_j B_j^\dagger X_j B_j\right)}\right]^{1/2}$$

where the supremum is taken over all choices of psd matrices X_j in dimension n_j.

A BL datum $(\mathbf{B},\mathbf{p})$ is called *geometric* if it satisfies the following conditions:

1. Projection: For every $j \in [m]$, B_j is a projection, namely $B_j B_j^\dagger = I_{n_j}$.

2. Isotropy: $\sum_j p_j B_j^\dagger B_j = I_n$.

Problem 4.4. *Using Lieb's description, prove that the BL constant is at least* 1 *if the BL datum satisfies either the projection or isotropy property.*

Problem 4.5 (BL constant for geometric datum). *Using Lieb's description of the BL constant, prove that the BL constant of a geometric BL datum is always* 1.

Observe the close connection to Problem 3.5.

A positive operator is a linear map $T : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ which preserves the psd property i.e. $T(X) \succeq 0$ if $X \succeq 0$. The capacity of a positive operator is defined as follows [Gur04]:

$$\mathrm{cap}(T) = \inf_{X \succ 0 : \mathrm{Det}(X)=1} \mathrm{Det}(T(X))$$

T is called rank non-decreasing if $\mathrm{rank}(T(X)) \geq \mathrm{rank}(X)$ for all $X \succeq 0$.

Given an $n \times n$ unitary matrix U, let $u_1, \ldots, u_n$ denote its column vectors. Given a positive operator T, consider the non-negative matrix $A_{U,V}$ whose $(i,j)^{\mathrm{th}}$ entry is given by $\mathrm{tr}\left[T(u_i u_i^\dagger) v_j v_j^\dagger\right]$. So this describes a reduction from operators to non-negative matrices after fixing a basis.

Problem 4.6. *Use the above described reduction to prove that* $\mathrm{cap}(T) > 0$ *iff* T *is rank non-decreasing.*

This is quite common in geometric invariant theory, fixing basis and compactness arguments reduce the non-commutative group actions to the setting of commutative group actions. However this reduction is highly non-algorithmic and finding the "right basis" is one of the main challenges in the algorithmic setting.

A positive operator T is called *completely positive* if there are matrices $A_1, \ldots, A_m \in M_n(\mathbb{C})$ such that

$$T(X) = \sum_{i=1}^{m} A_i X A_i^\dagger, \quad \forall X \in M_n(\mathbb{C}).$$

Problem 4.7. *Prove that the problem of computing the rank of a linear matroid (or of the intersection of two linear matroids) can be reduced to the problem of determining whether a completely positive operator is rank non-decreasing.*

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