Complete Derandomization of Identity Testing of Read-Once Formulas

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Outline

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- Method of representation
- Efficiency
- Allowed actions

Succinct Representation

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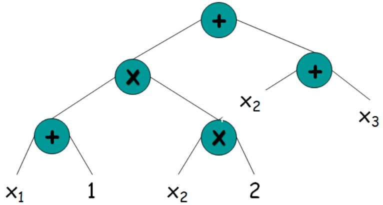
Definition

An arithmetic formula is a directed tree from variables $(x_1, \dots, x_n)$ to an output. Each leaf is assigned a variable or field element and each internal node, or gate, is assigned an operation $+$ or $\times$.

Idea: Model small computational devices

Figure: Example of an Arithmetic Formula for $(x_1 + 1)(2x_2) + x_2 + x_3$



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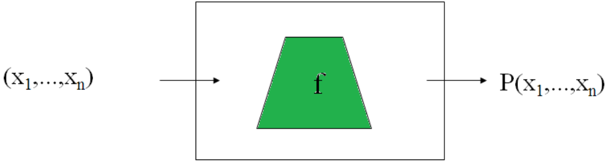

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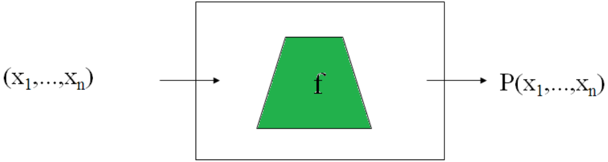

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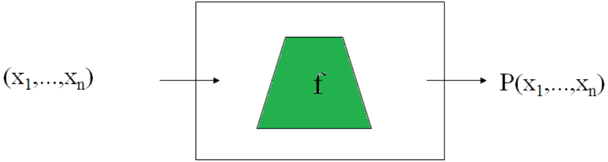

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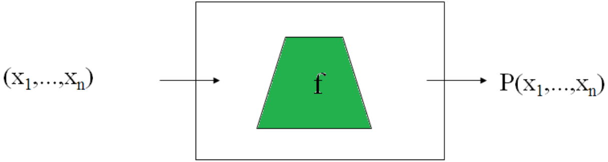

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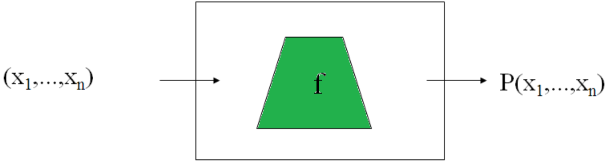

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- $\mathcal{H}$ gives an algorithm that runs in time $O(|\mathcal{H}|)$.
- Goal: find a small hitting set for polynomials computed by small formulas.

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Let $P \in \mathbb{F}[x_1, \ldots, x_n]$ with total degree bounded by some $d \in \mathbb{N}$. Pick some finite $S \subseteq \mathbb{F}$. Then $\Pr_{\bar{x} \in S^n}[P(\bar{x}) = 0] \leq \frac{d}{|S|}$.

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- This does not provide a deterministic algorithm but suggests that one might exist.

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I have a truly marvelous proof of these lower bounds which this slide is too small to contain.

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- Bounded-read formulas:
  [ASS12, AvMV15, SV15, FS13, AFS^{+}16]

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Definition (Read-Once Polynomial)

A polynomial that can be expressed by an arithmetic formula f such that no variable appears more than once. For example, $x_1x_2 + x_1x_3$ but not $x_1x_2 + x_2x_3 + x_3x_1$.

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Theorem (Our Result)

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Results and Implications

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Theorem (Corollary 2 from [SV15])

For every $k \in \mathbb{N}$ there exists an explicit hitting set of size $n^{O(k)}$ for sums of k ROPs.

Approach

Definition (Generator)

Pick some $k \in \mathbb{N}$ with $k \ll n$. Let $\mathcal{C} \subseteq \mathbb{F}[x_1, \ldots, x_n]$. A *generator* for $\mathcal{C}$ is a polynomial map $G : \mathbb{F}^k \to \mathbb{F}^n$ such that $\forall P \in \mathcal{C}$, we have $P(G) \equiv 0 \iff P \equiv 0$.

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Let $P \in \mathbb{F}[x_1, \ldots, x_n]$ where the degree of each variable in P is bounded by some $d \in \mathbb{N}$. For any set $S \subseteq \mathbb{F}$ of size at least $d+1$, $\exists \bar{a} \in S^n$ such that $P(\bar{a}) \neq 0$.

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Lemma (From Generator to Hitting Set)

Suppose the individual degrees of each component function of G are bounded by some $d \in \mathbb{N}$. Then we can decide if $P(G) \equiv 0$ in time $(nd)^{O(k)}$.

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Definition (The Generator of [SV09])

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Theorem (Our Result - Polynomial PIT)

The polynomial map $G_{n,\mathbf{1}}$ is a generator for ROPs on n variables.

High Level Idea

Definition (Homogenous Polynomial)

A polynomial $P \in \mathbb{F}[x_1, \ldots, x_n]$ is called *homogeneous* if every term in the polynomial has the same total degree.

Lemma (Generator for Homogeneous ROPs)

$G_{n,1}$ is a generator for homogeneous ROPs.

Lemma (From Homogeneous ROPs to General ROPs)

If $P \in \mathbb{F}[x_1, \ldots, x_n]$ is a ROP, then so is $H_{\deg(P)}(P)$.

Corollary (From Homogeneous ROPs to ROPs)

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Lemma (Homogenous ROP Structural Lemma)

If $P \in \mathbb{F}[x_1, \ldots, x_n]$ is a homogeneous ROP with $n \geq 2$, then $\exists P_1, P_2$ non-constant, variable-disjoint homogeneous ROPs s.t:

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0. Linear Case (Base case): $P = c_1x_1 + \cdots + c_nx_n$.

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Pick $\{\alpha_1, \ldots, \alpha_n\} \subseteq \mathbb{F}$. Let $\mu_i(y)$ the i^{th} Lagrange Interpolation Polynomial. Let $G_{n,1}(y, z) = (\mu_1(y)z, \ldots, \mu_n(y)z)$.

Lemma (Homogenous ROP Structural Lemma)

If $P \in \mathbb{F}[x_1, \ldots, x_n]$ is a homogeneous ROP with $n \geq 2$, then $\exists P_1, P_2$ non-constant, variable-disjoint homogeneous ROPs s.t:

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2. $P = P_1 + P_2$

By induction on n:

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Pick $\{\alpha_1, \dots, \alpha_n\} \subseteq \mathbb{F}$. Let $\mu_i(y)$ the i^{th} Lagrange Interpolation Polynomial. Let $G_{n,1}(y,z) = (\mu_1(y)z, \dots, \mu_n(y)z)$.

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2. Additive Case: Main technical contribution of the paper.

Conclusion & Open Questions

Theorem (Black-Box PIT for ROFs)

There exists a polynomial-time black-box PIT algorithm for read-once formulas.

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- Polynomial-time black-box PIT algorithm for read-twice formulas?

Thank you!

Thank you!

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