Exponential lower bounds for hom. depth-5 circuits over finite fields

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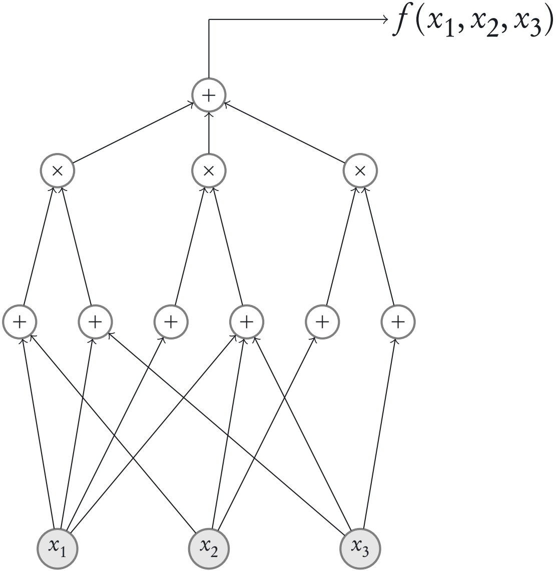
CCC 2017
Riga

Algebraic Circuits



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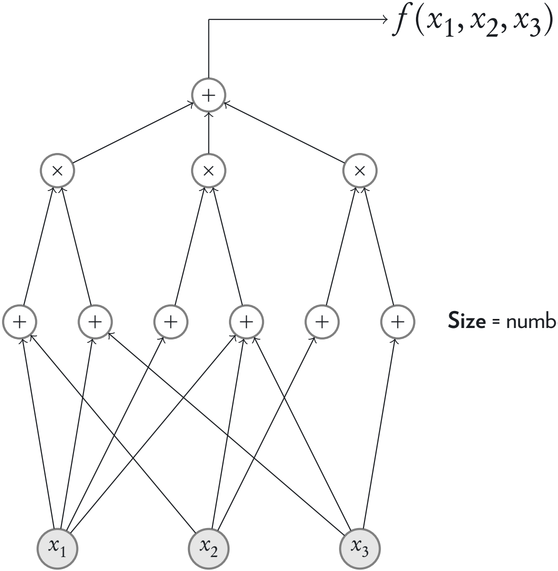
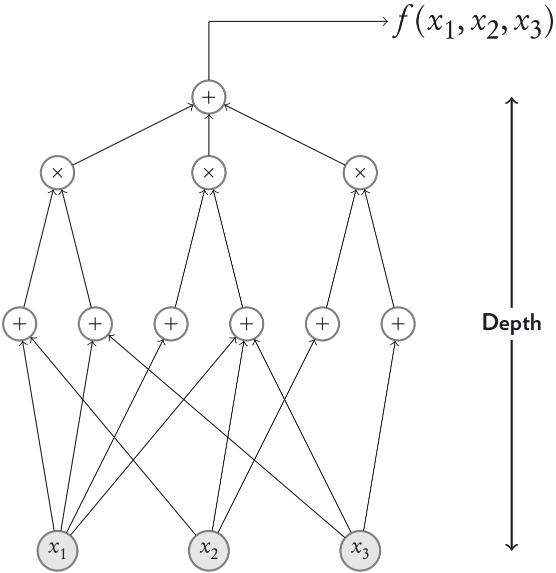


Size = number of gates

Algebraic Circuits



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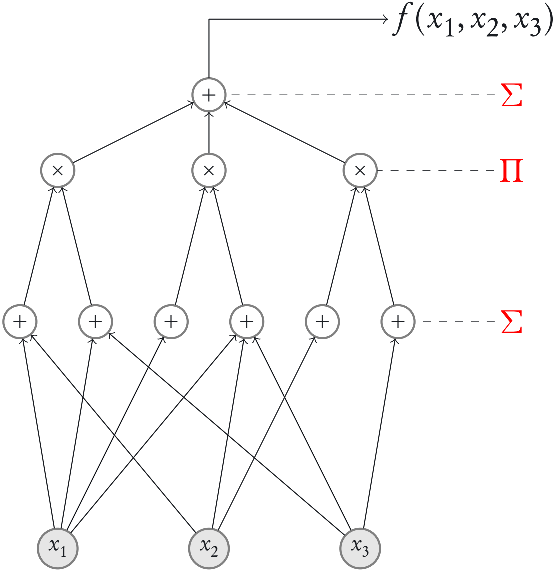

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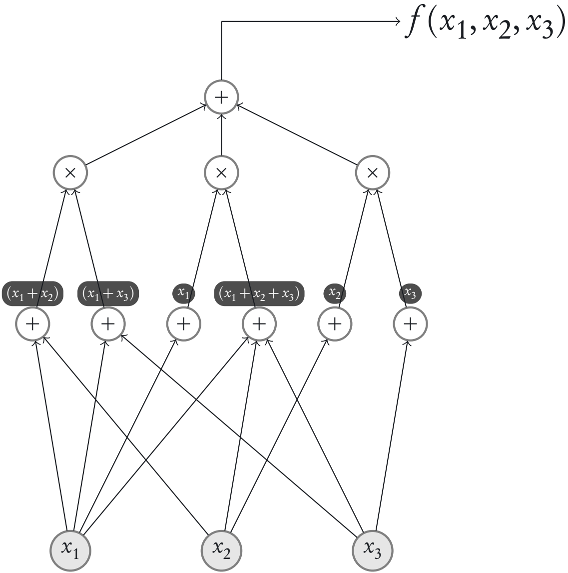

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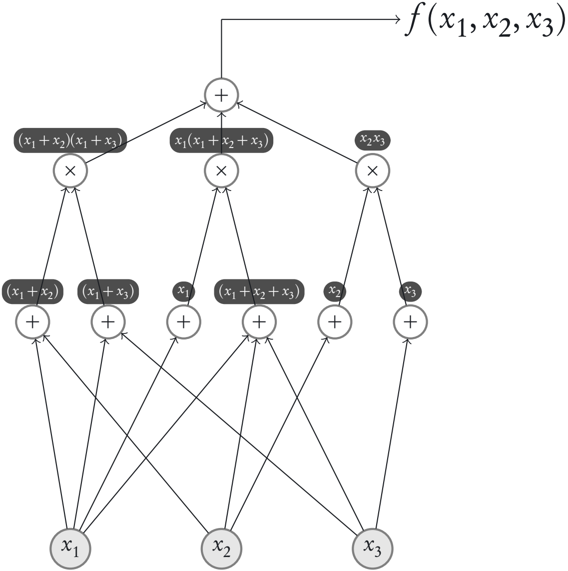

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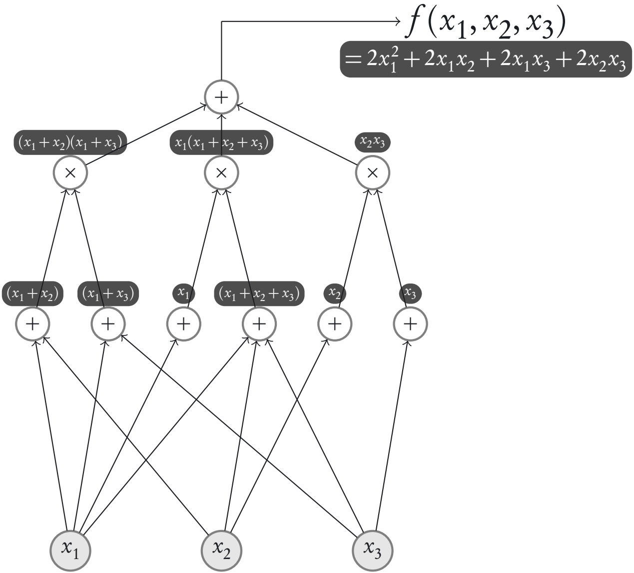

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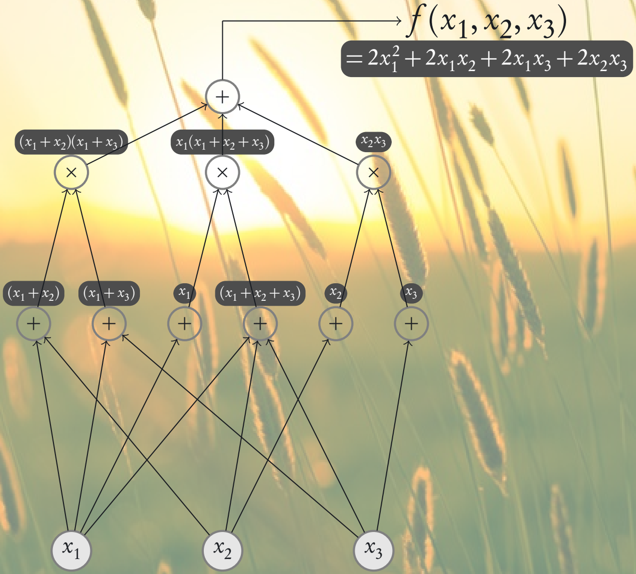

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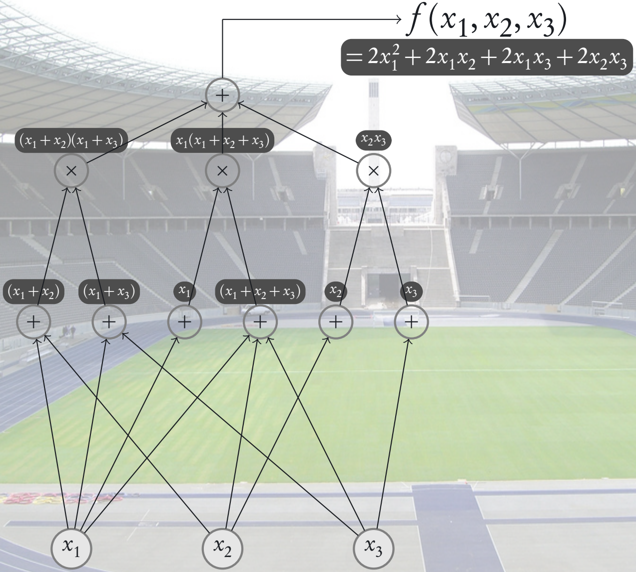

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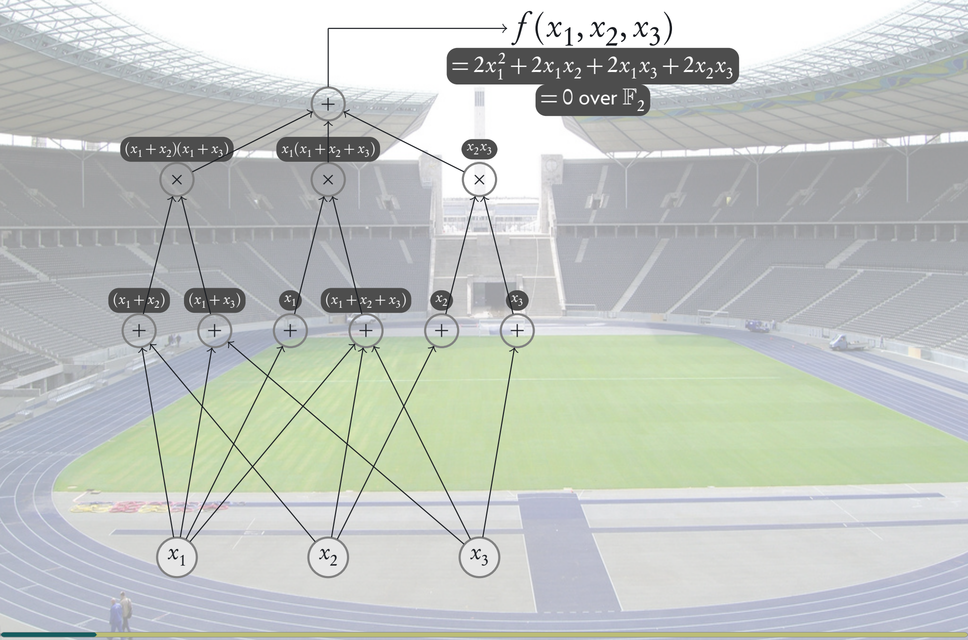

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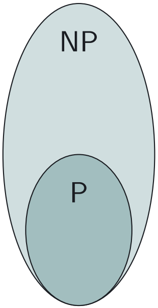

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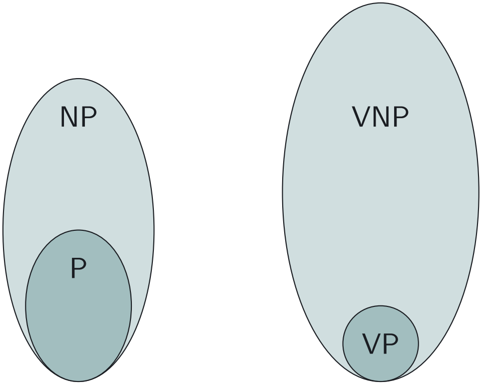

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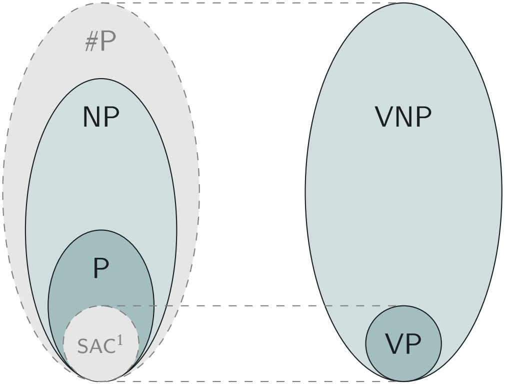

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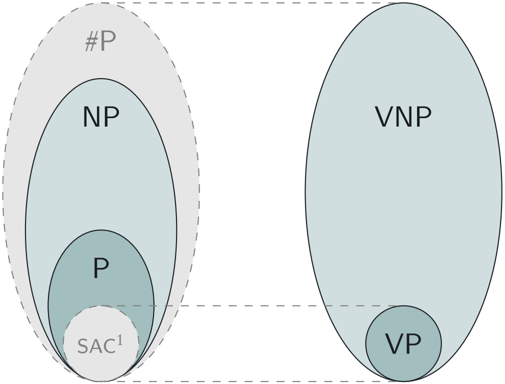

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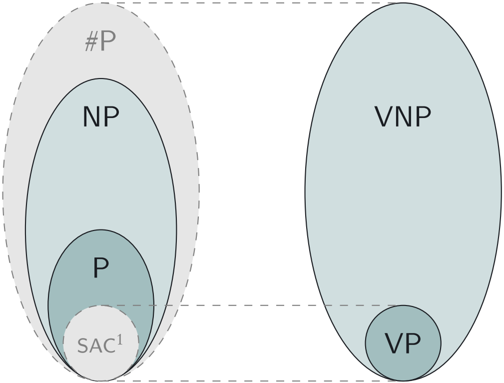

VP $\neq$ VNP is simpler to prove than P $\neq$ NP.

Ultimate goal: Find an explicit n-variate degree d polynomial that requires large arithmetic circuits to compute it.

Depth Reduction

Theorem ([Agrawal-Vinay + Koiran, Tavenas])

Can be computed by		*Can be computed by*
algebraic circuits	$\longrightarrow$	*depth-4 circuits*
of "small" size		*of "not-too-large" size*

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Goal: To prove an $n^{\omega(\sqrt{d})}$ lower bound for $\Sigma\Pi^{[\sqrt{d}]}\Sigma\Pi^{[\sqrt{d}]}$ circuits.

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Theorem ([Kayal-Limaye-Saha-Srinivasan])
A $n^{\Omega(\sqrt{d})}$ lower bound for homogeneous depth-4 circuits.

Our results

Theorem

An explicit polynomial $f(x_1, \ldots, x_n)$ of degree d with $0/1$ coefficients such that, for any fixed finite field $\mathbb{F}_q$, any homogeneous $\Sigma\Pi\Sigma\Pi\Sigma$ circuit computing f must have size $2^{\Omega_q(\sqrt{d})}$.

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... ought to have been easier than this

How are such bounds proved?

Natural proof strategies

Construct a map $\Gamma : \mathbb{F}[x_1, \ldots, x_n] \to \mathbb{N}$, that assigns a number to every polynomial such that:

1. If f is computable by "small" circuits, then $\Gamma(f)$ is "small".
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  Therefore, if $\text{Det}_d = \sum_{i=1}^{s} \ell_{i1} \cdots \ell_{id}$, then $s \geq \binom{d}{d/2}$. $\square$

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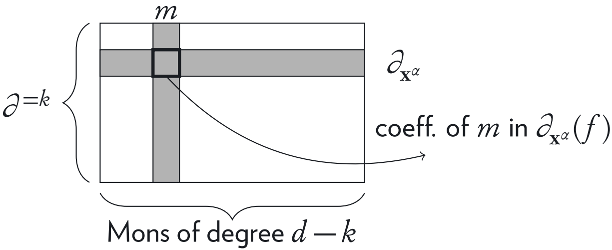

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$$\Gamma(f) = \dim \left\{ \mathbf{x}^{=\ell} \partial^{=k}(f) \right\}$$

Dimension of shifted partial derivatives

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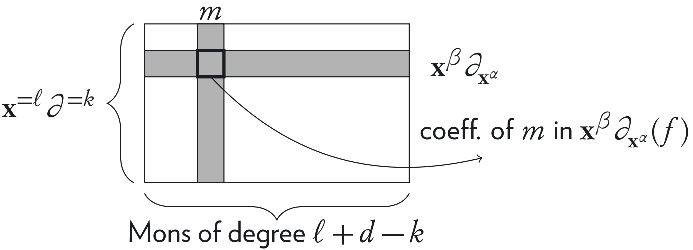

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- [Kayal-Limaye-Saha-Srinivasan-13], [Kumar-Saraf-13]: hom. $\Sigma\Pi\Sigma\Pi$ circuits: terms like $Q_1 \cdots Q_b$ with total degree d.

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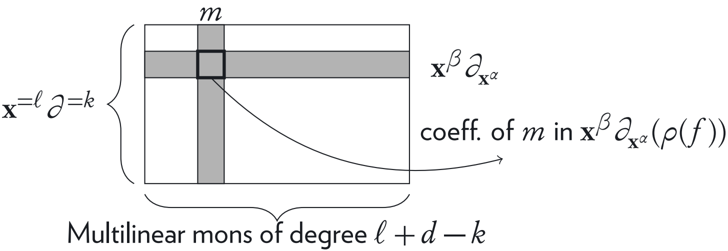

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... still don't know

Evaluating the complexity measure

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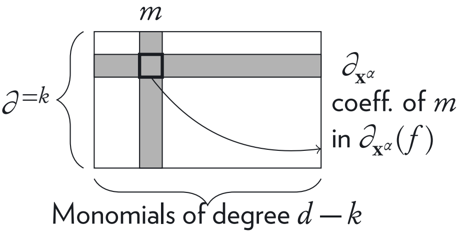

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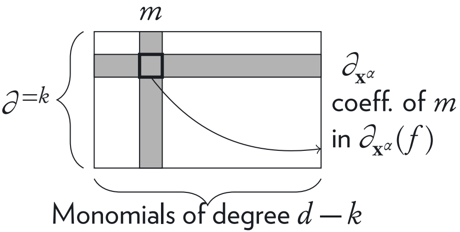

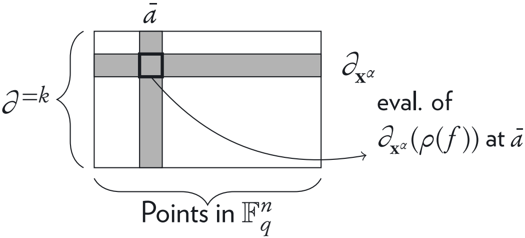

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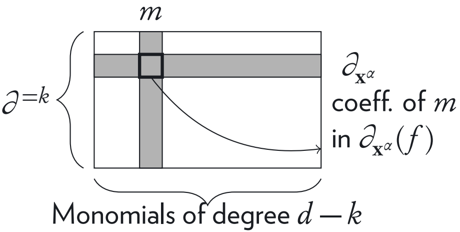
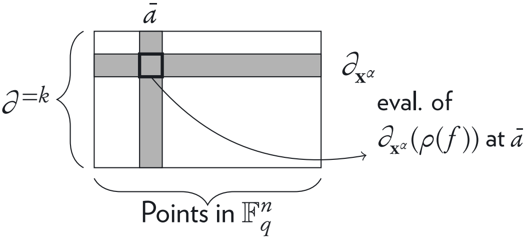


Small rank

$\Downarrow$



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Low degree terms.	High degree high rank terms	High degree low rank terms

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If $\dim \{\ell_1, \cdots, \ell_r\}$ is large, then almost all evaluations of it on $\mathbb{F}_q^n$ are zero.

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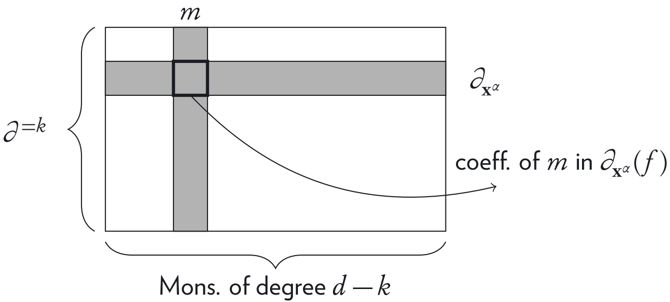

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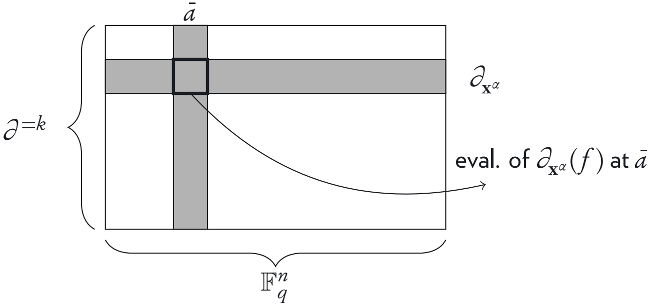

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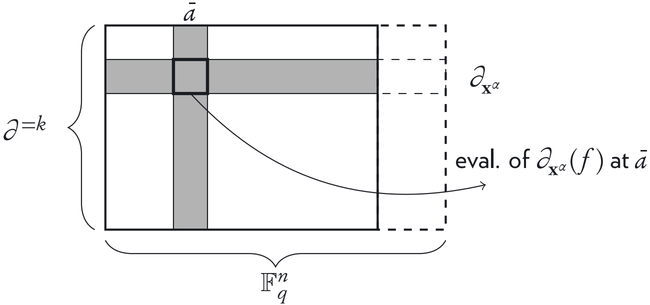

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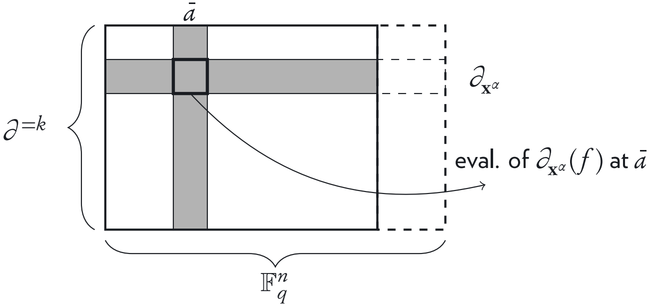

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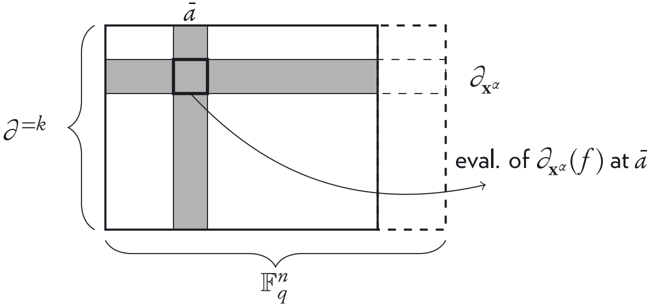

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For Det_n or Perm_n the above matrix remains full rank, as long as we removed only few columns.

Lifting to depth five

$\Sigma\Pi\Sigma\Pi\Sigma$

Types of products of linear polynomials:

Low degree products.	High degree products.

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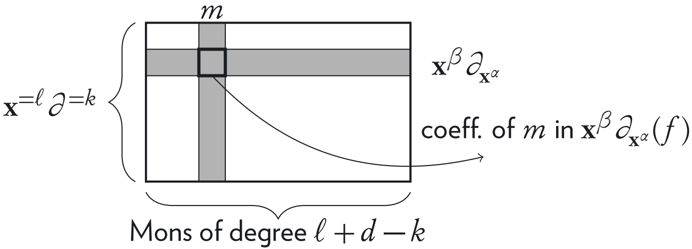

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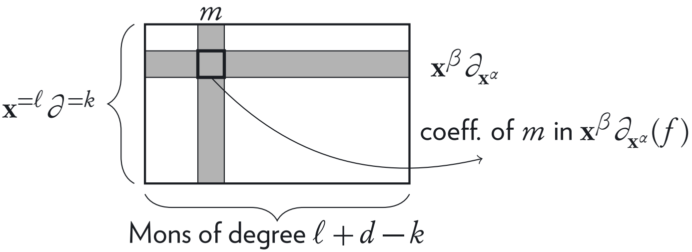
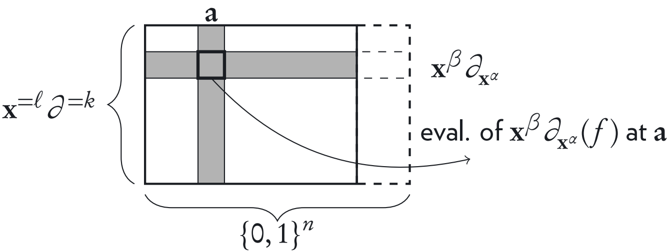


NEED TO SHOW *this* RANK IS LARGE:



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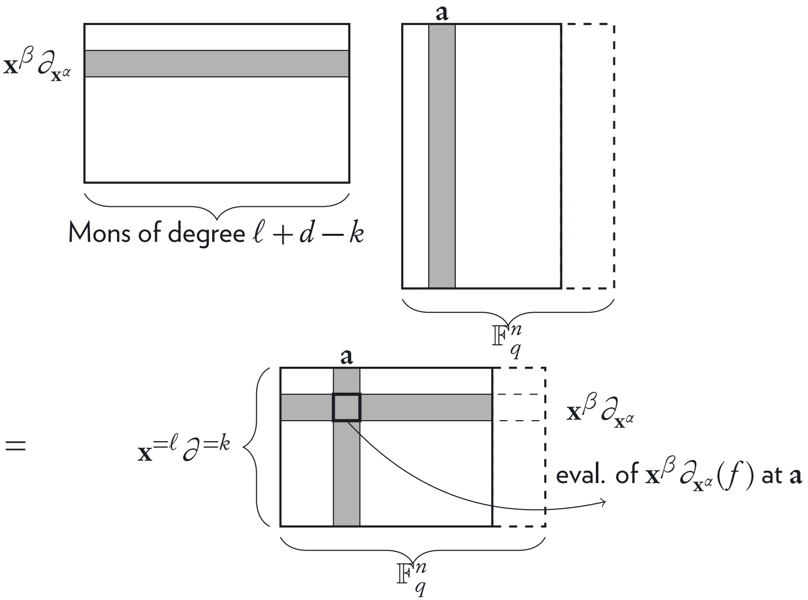

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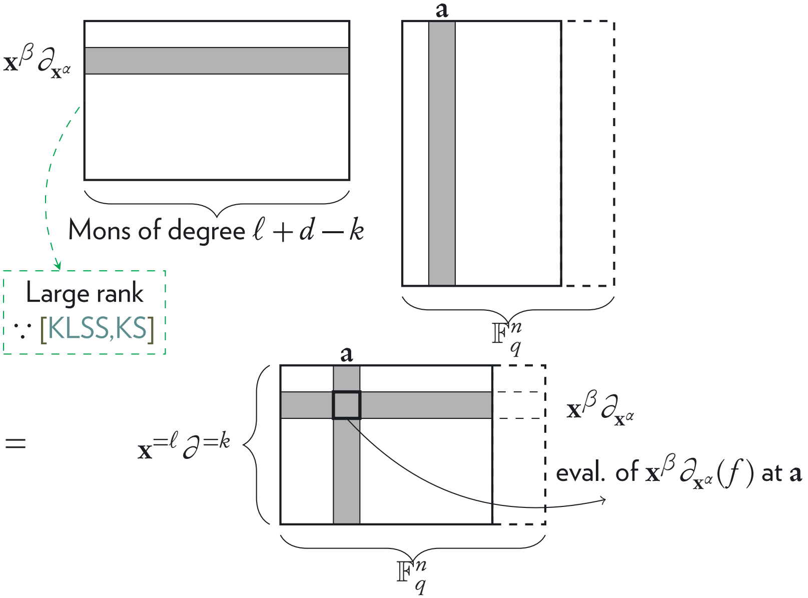

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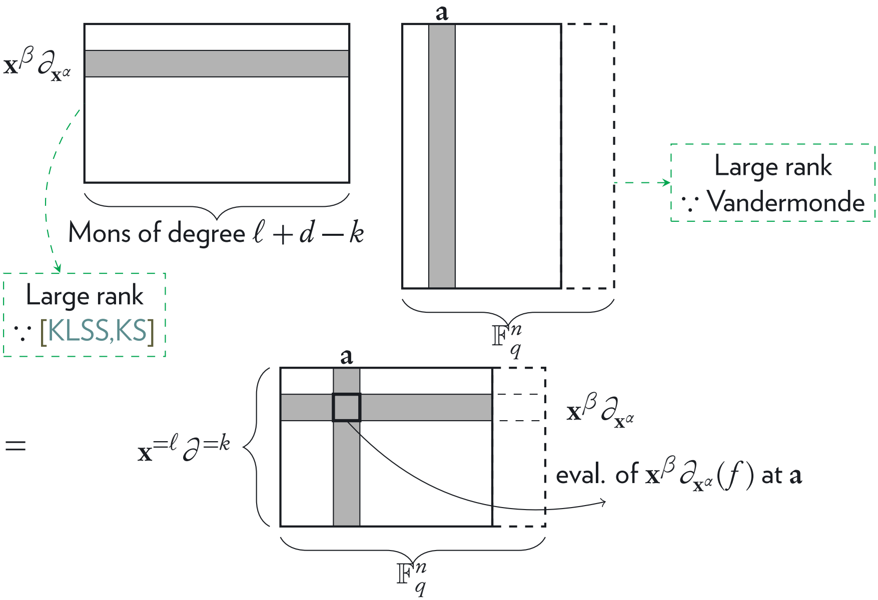

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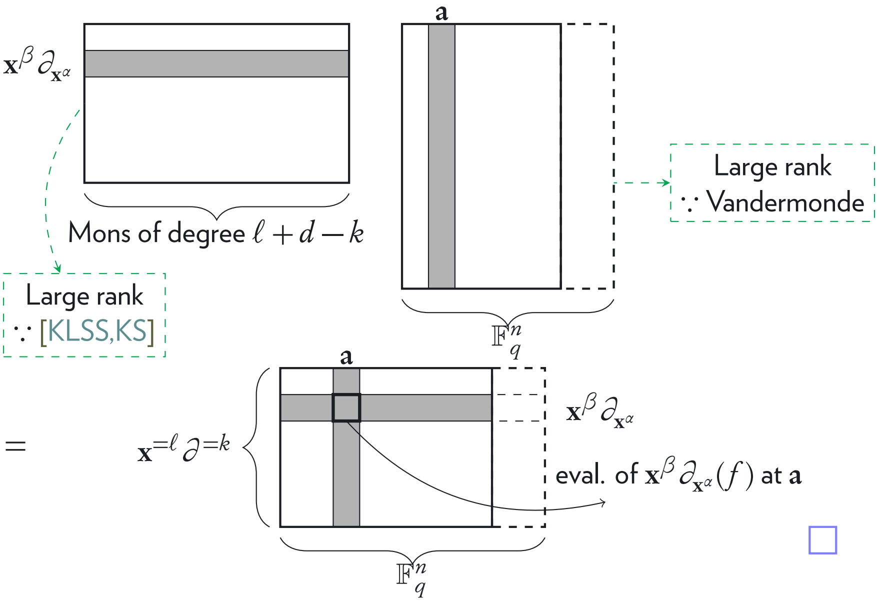

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(Barely manages to work for a specific explicit polynomial. Phew!) $\square$

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Theorem

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\end{document}

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