Exponential lower bounds for hom. depth-5 circuits over finite fields

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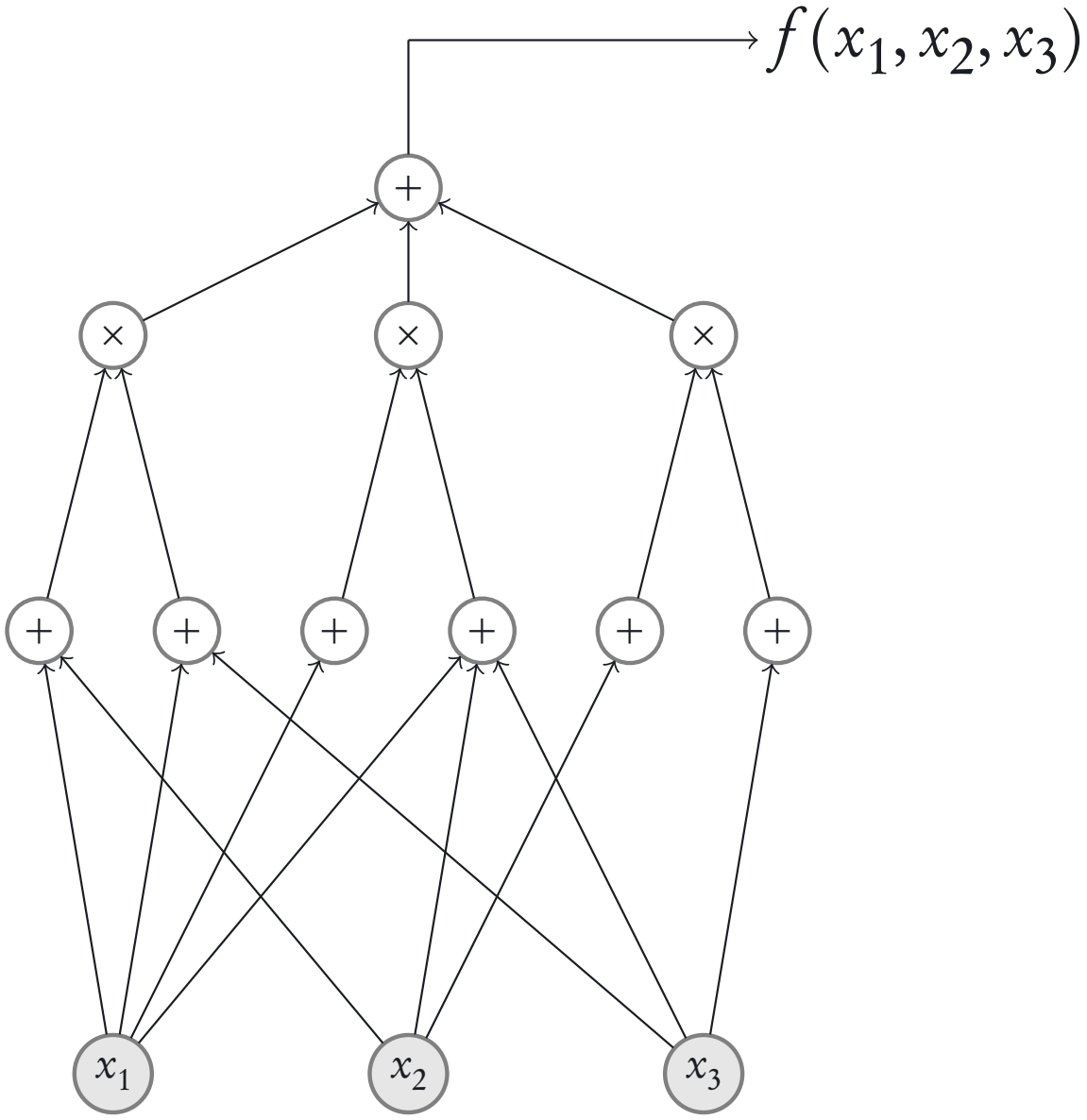
CCC 2017
Riga

Algebraic Circuits



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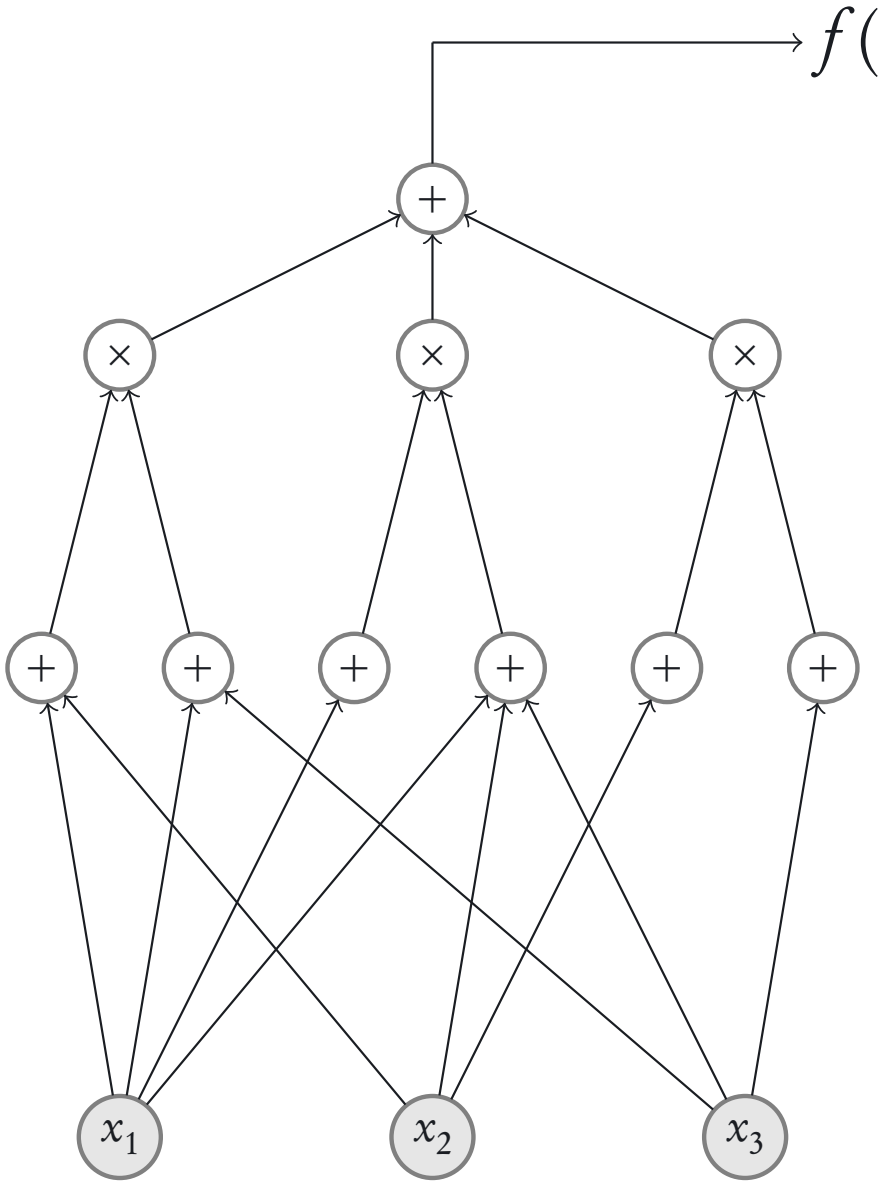
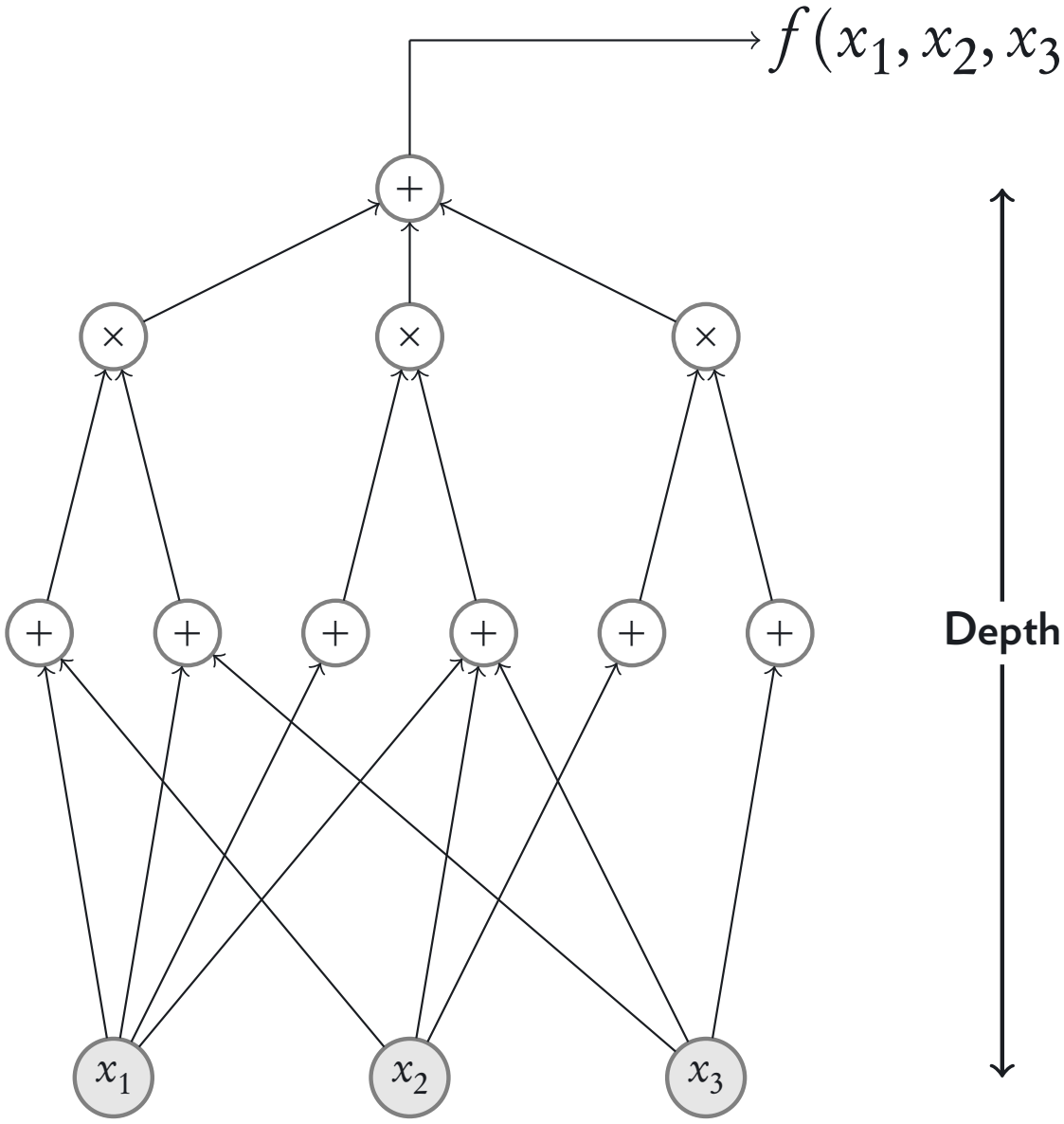


Size = number of gates

Algebraic Circuits



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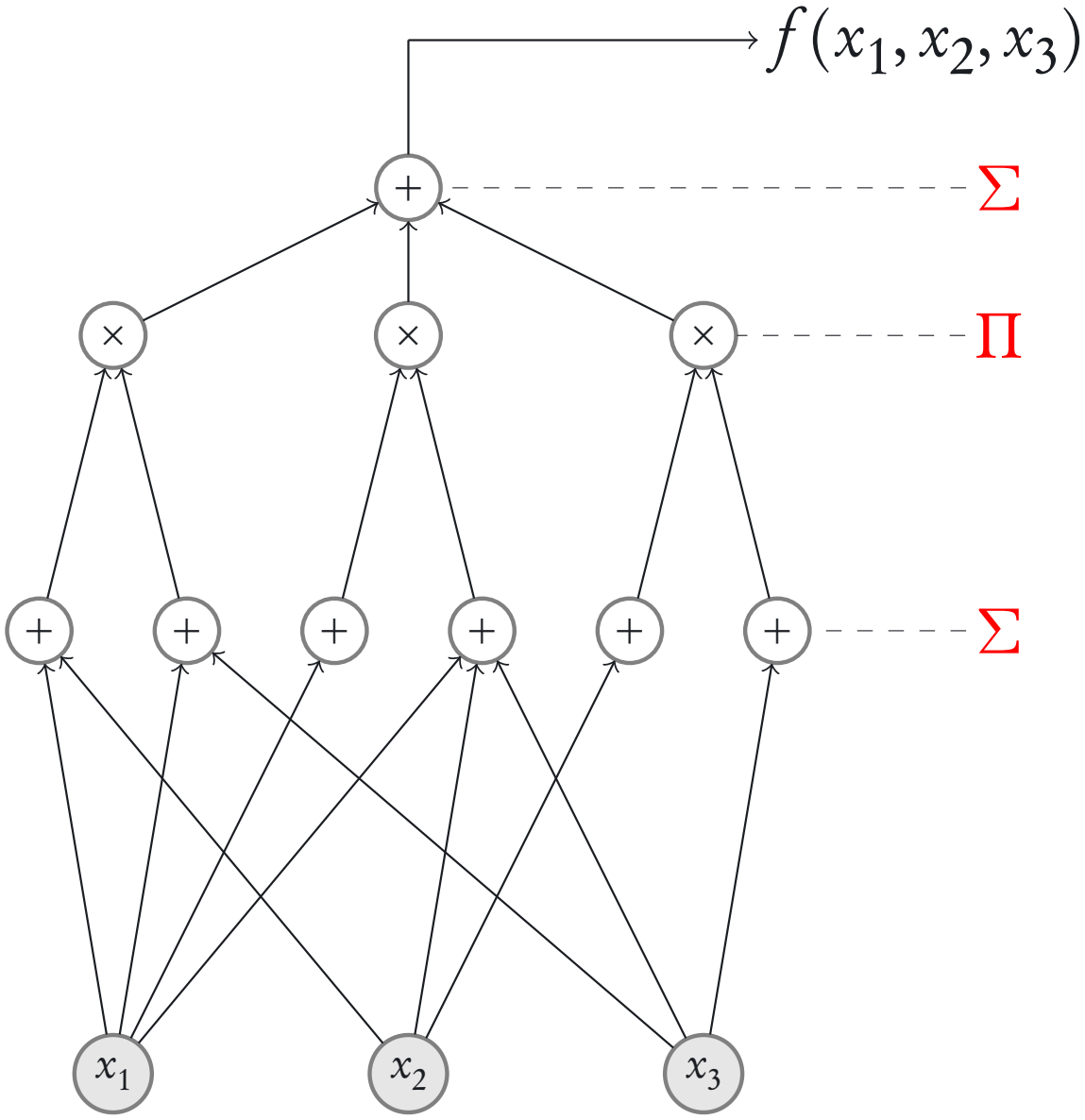

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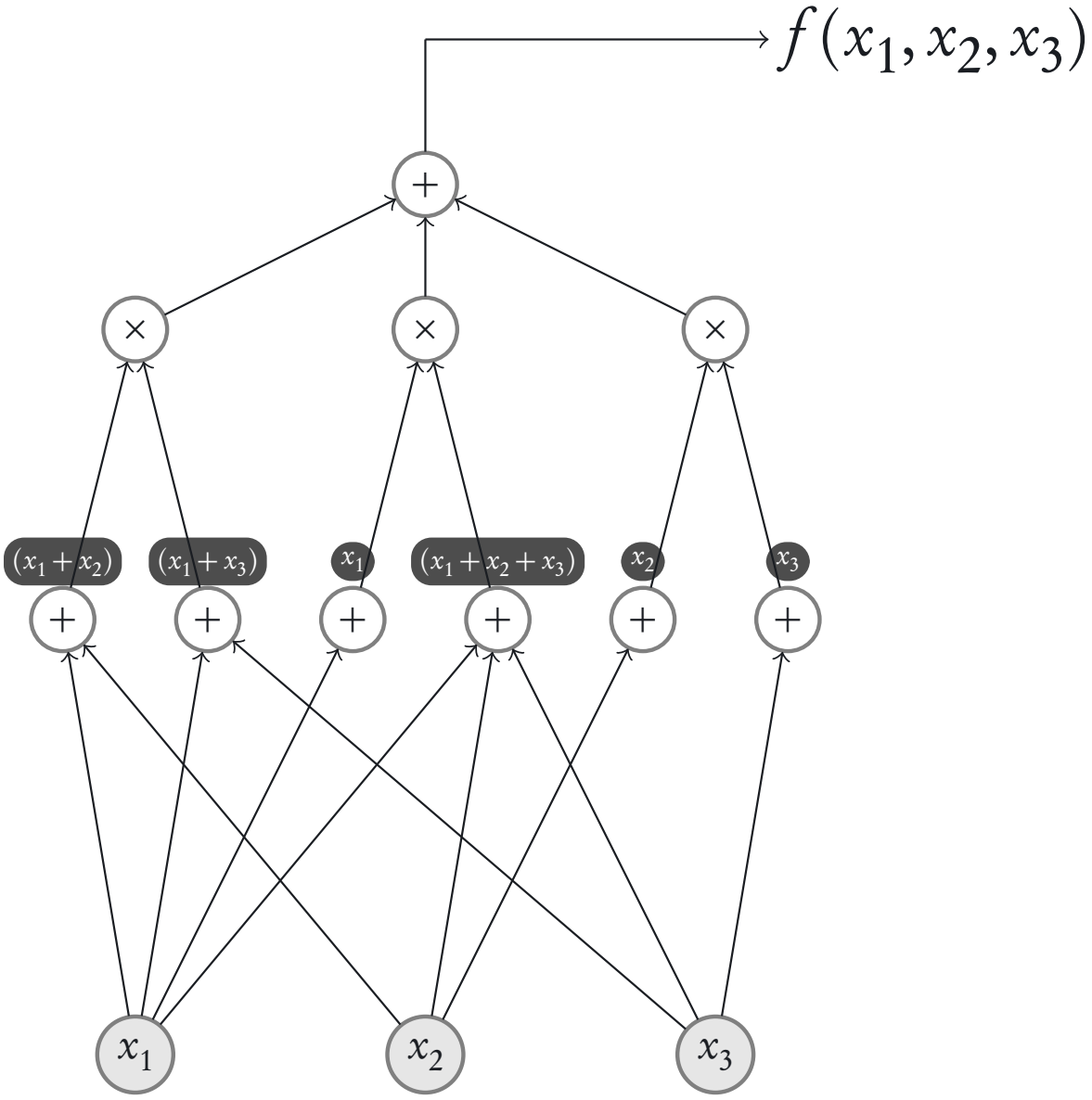

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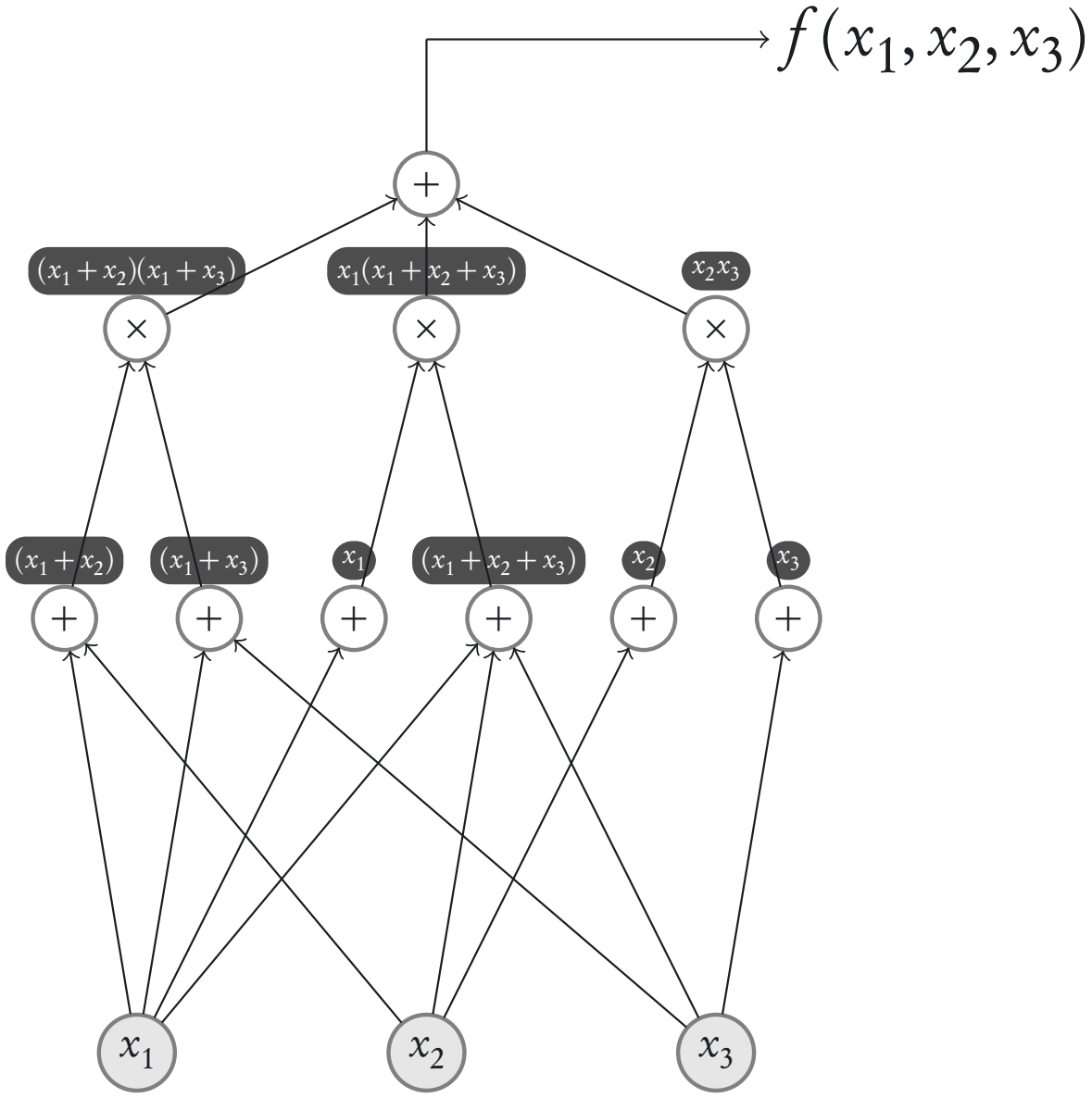

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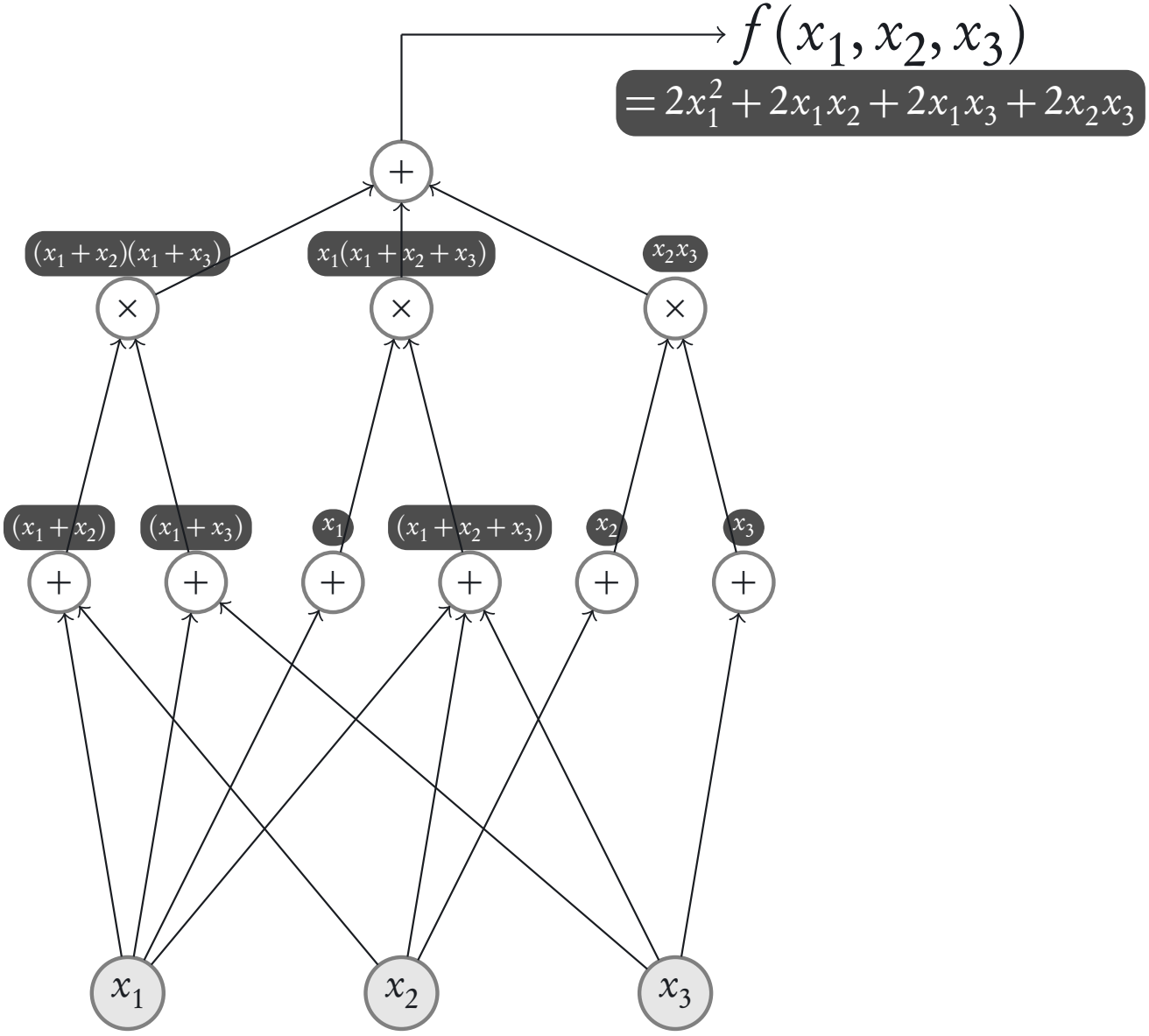

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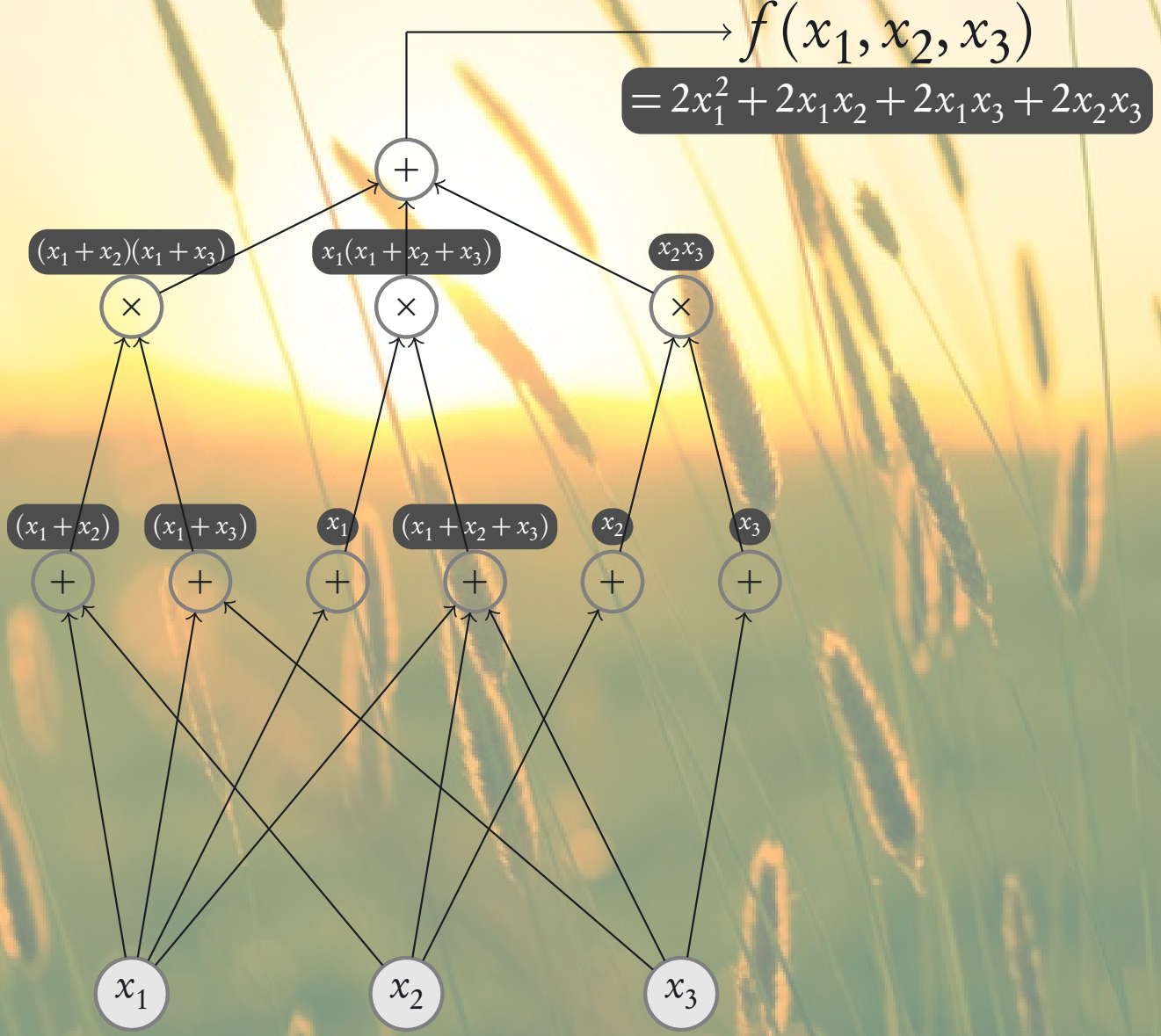

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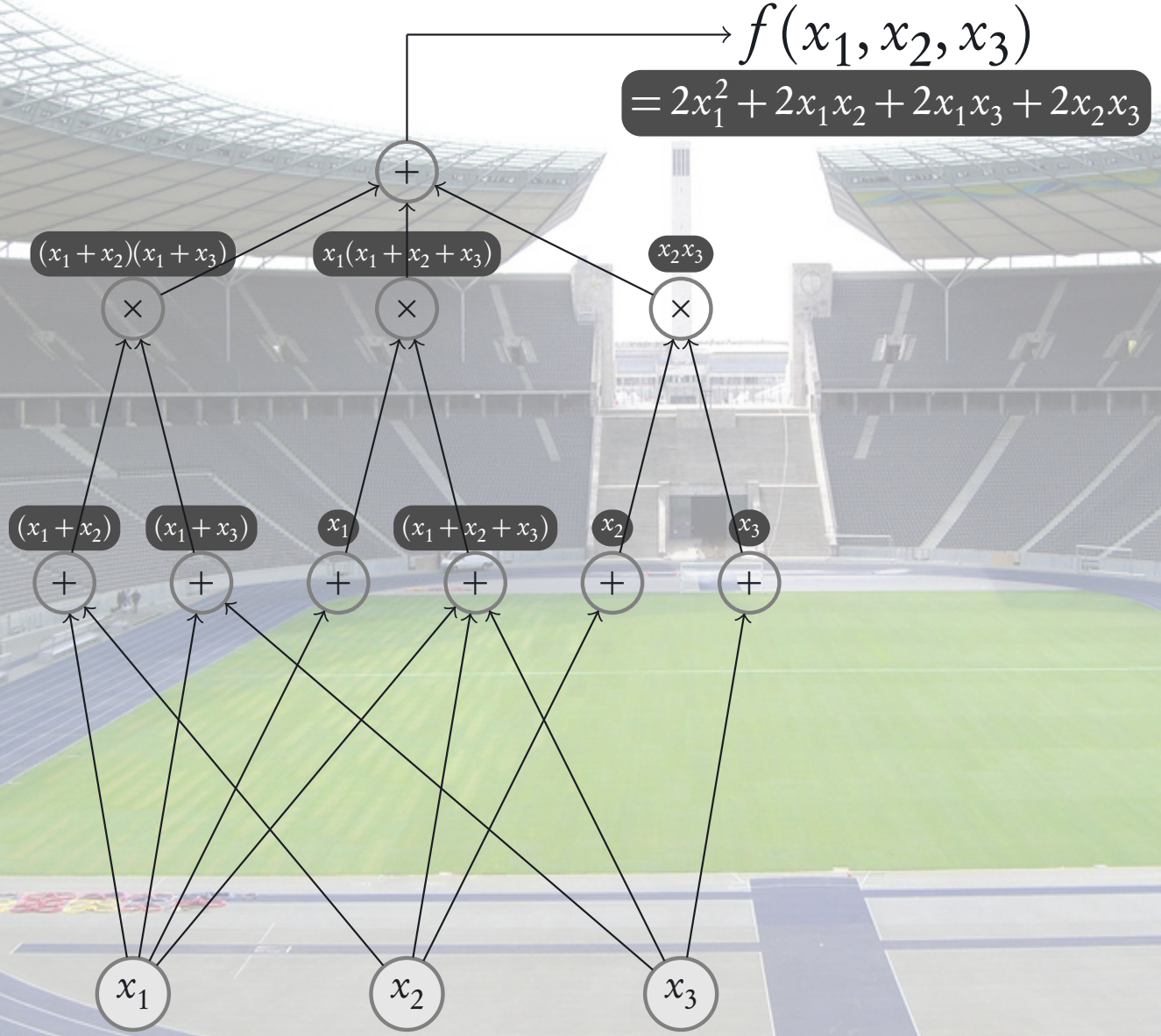

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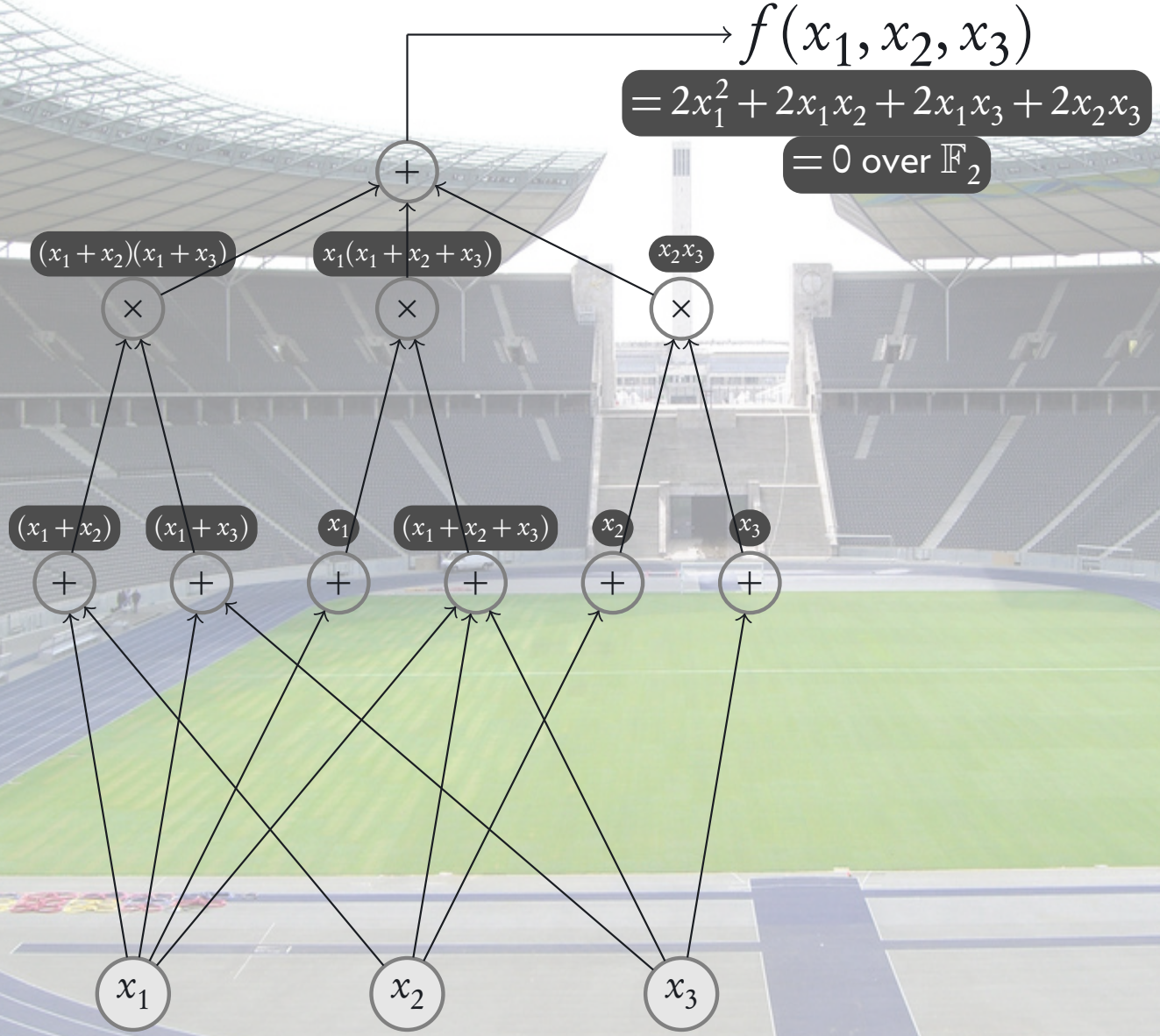
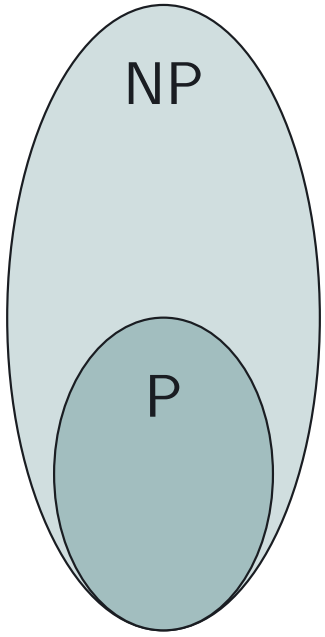


The Open Problem(s)



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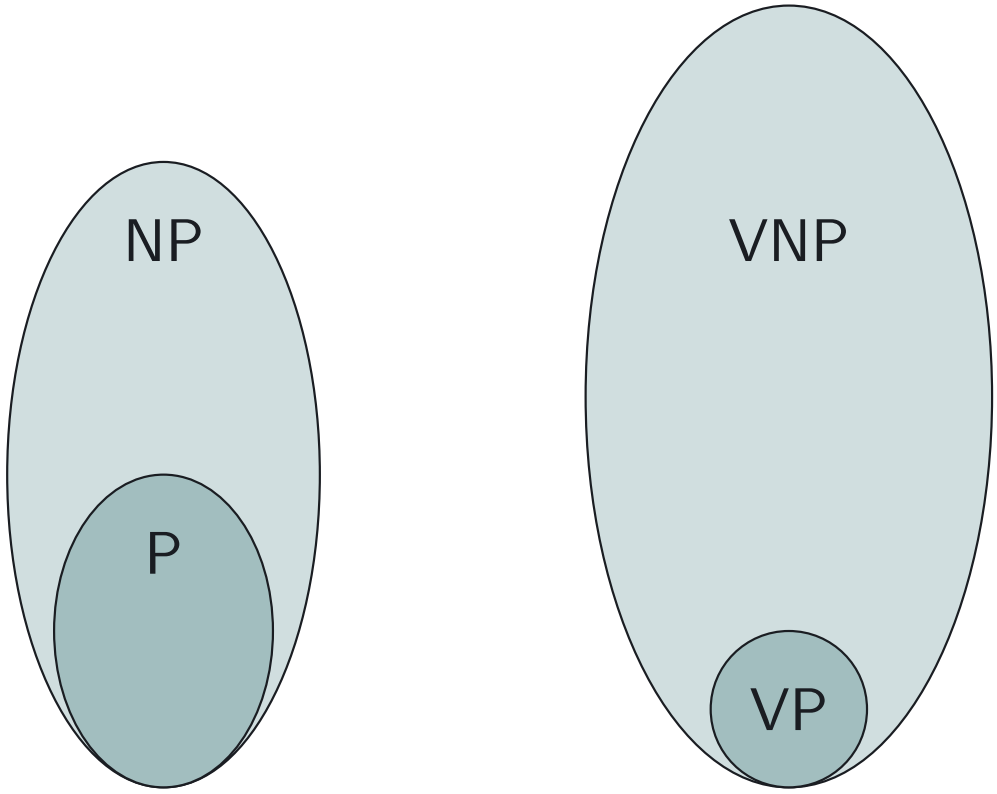

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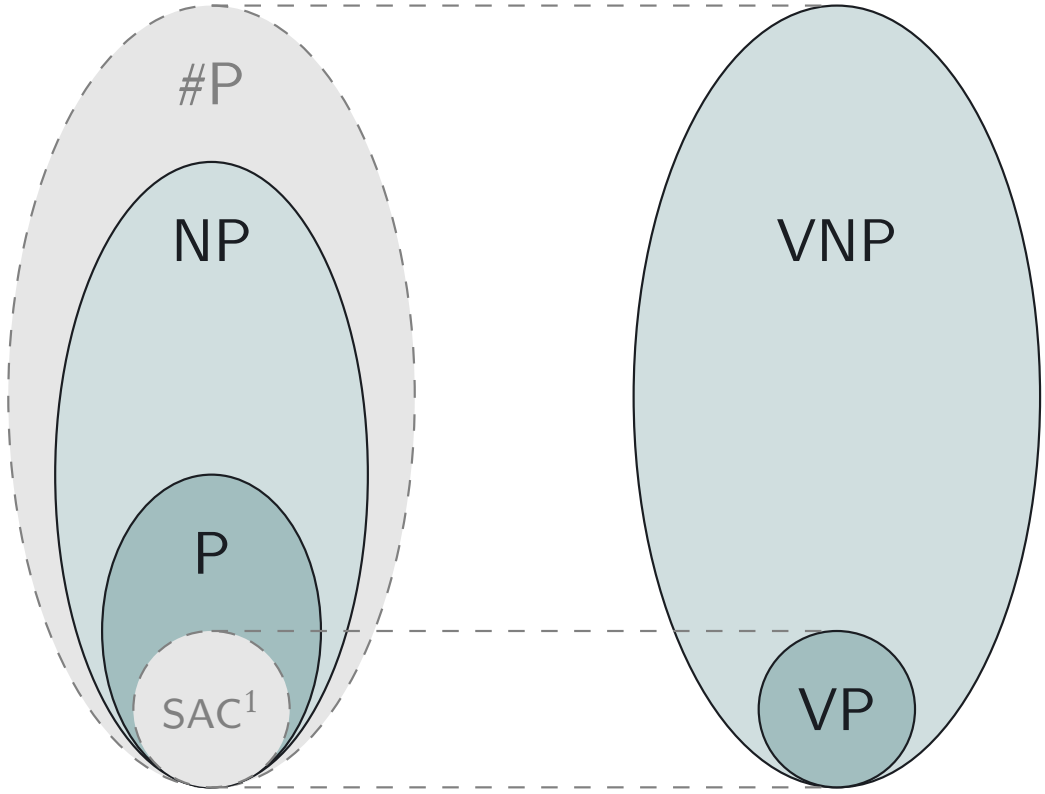

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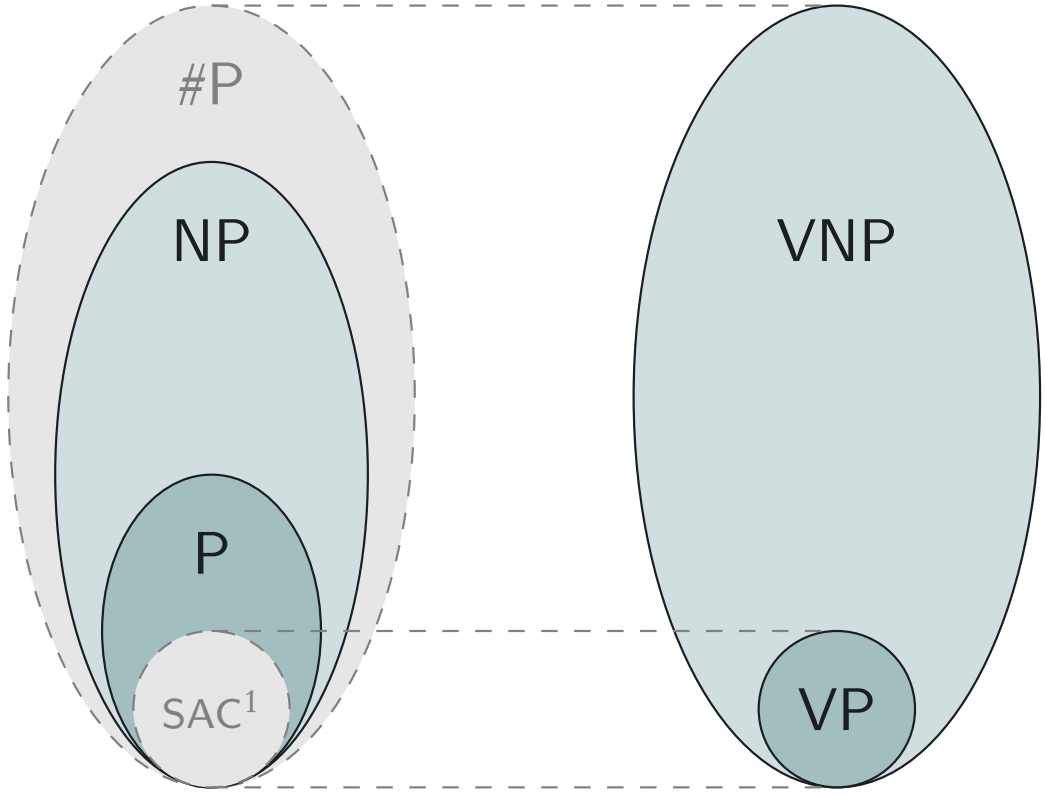

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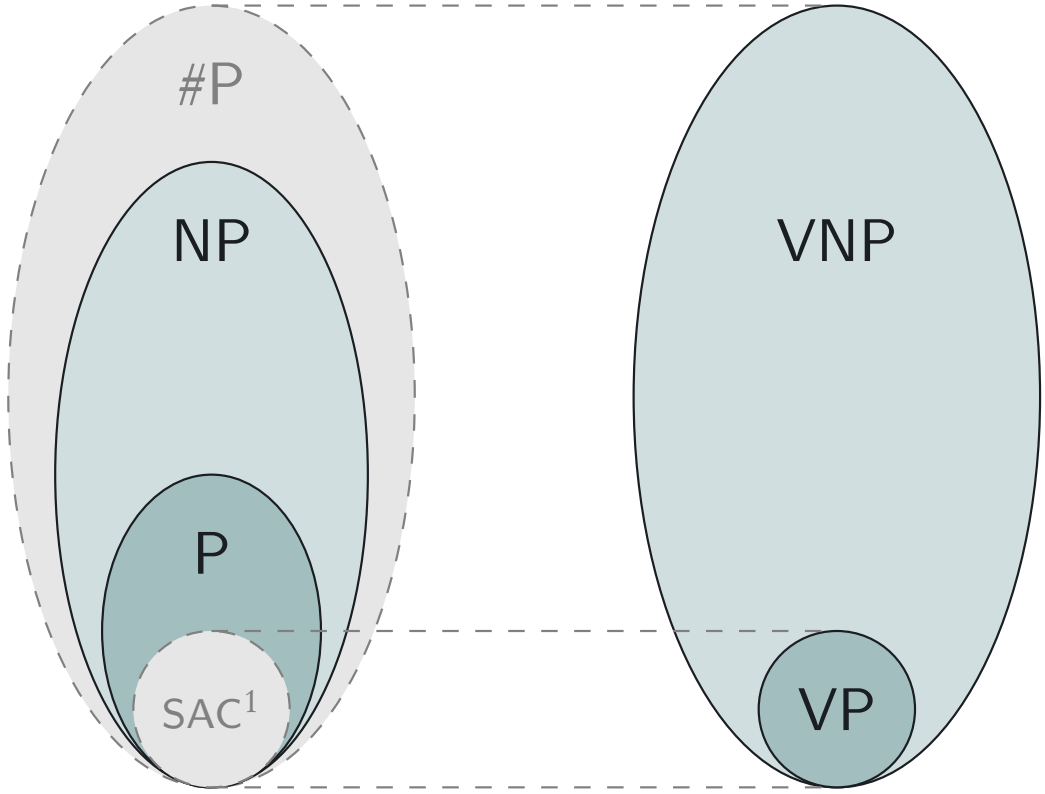

VP $\neq$ VNP is simpler to prove than P $\neq$ NP.

Ultimate goal: Find an explicit n-variate degree d polynomial that requires large arithmetic circuits to compute it.

Depth Reduction

Theorem ([Agrawal-Vinay + Koiran, Tavenas])

Can be computed by

algebraic circuits

of "small" size

$\longrightarrow$

Can be computed by

depth-4 circuits

of "not-too-large" size

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Theorem ([Agrawal-Vinay + Koiran, Tavenas])

Can be computed by

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of $\mathrm{poly}(n, d)$ *size*

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Can be computed by

$\Sigma\Pi^{[\sqrt{d}]}\Sigma\Pi^{[\sqrt{d}]}$ *circuits*

of $n^{O(\sqrt{d})}$ *size*

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(Or)

Cannot be computed by

algebraic circuits $\longleftarrow$ $\Sigma\Pi^{[\sqrt{d}]}\Sigma\Pi^{[\sqrt{d}]}$ *circuits*

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Goal: To prove an $n^{\omega(\sqrt{d})}$ lower bound for $\Sigma\Pi^{[\sqrt{d}]}\Sigma\Pi^{[\sqrt{d}]}$ circuits.

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Theorem ([Kayal-Limaye-Saha-Srinivasan])

A $n^{\Omega(\sqrt{d})}$ lower bound for homogeneous depth-4 circuits.

Our results

Theorem

An explicit polynomial $f(x_1, \ldots, x_n)$ of degree d with $0/1$ coefficients such that, for any fixed finite field $\mathbb{F}_q$, any homogeneous $\Sigma\Pi\Sigma\Pi\Sigma$ circuit computing f must have size $2^{\Omega_q(\sqrt{d})}$.

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How are such bounds proved?

Natural proof strategies

Construct a map $\Gamma : \mathbb{F}[x_1, \ldots, x_n] \to \mathbb{N}$, that assigns a number to every polynomial such that:

1. If f is computable by "small" circuits, then $\Gamma(f)$ is "small".

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  Therefore, if $\mathrm{Det}_d = \sum_{i=1}^{s} \ell_{i1} \cdots \ell_{id}$, then $s \geq \binom{d}{d/2}$. □

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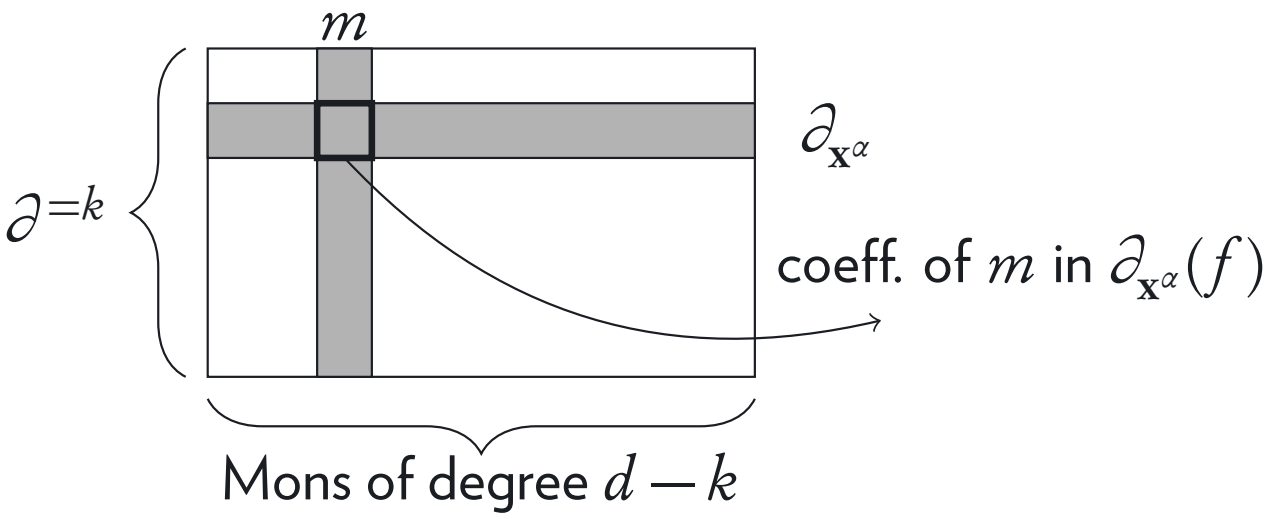

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$$\Gamma(f) = \dim\left\{\mathbf{x}^{=\ell}\,\partial^{=k}(f)\right\}$$

Dimension of shifted partial derivatives

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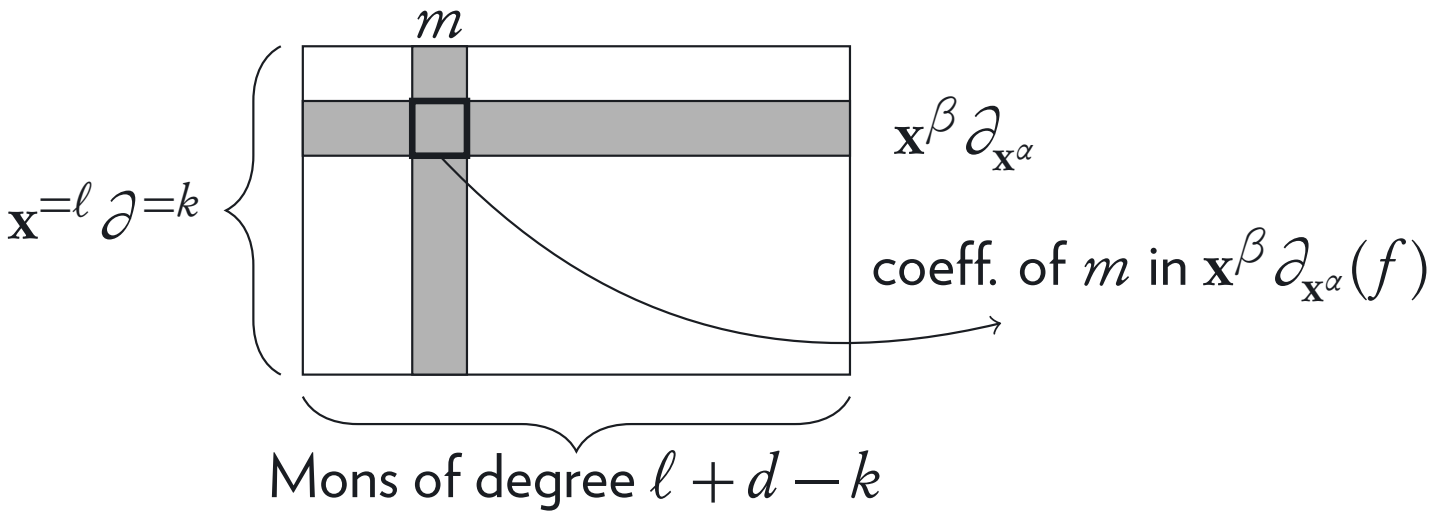

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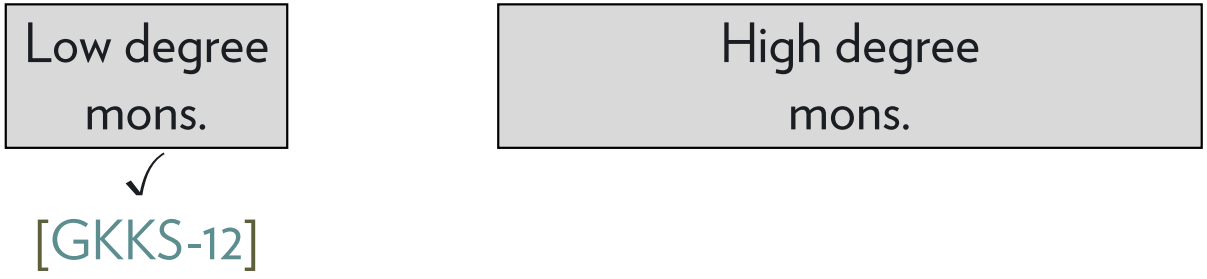

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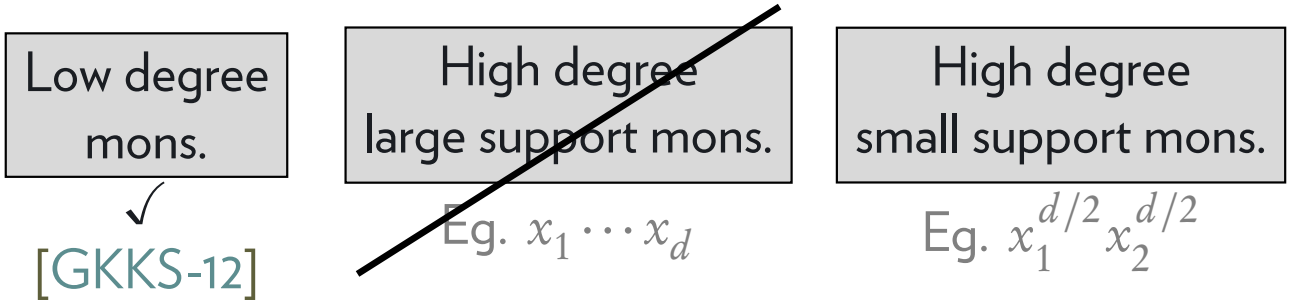

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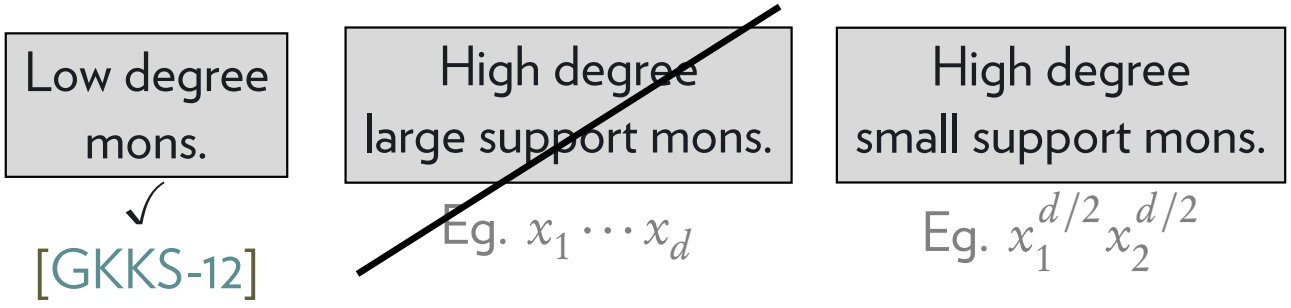

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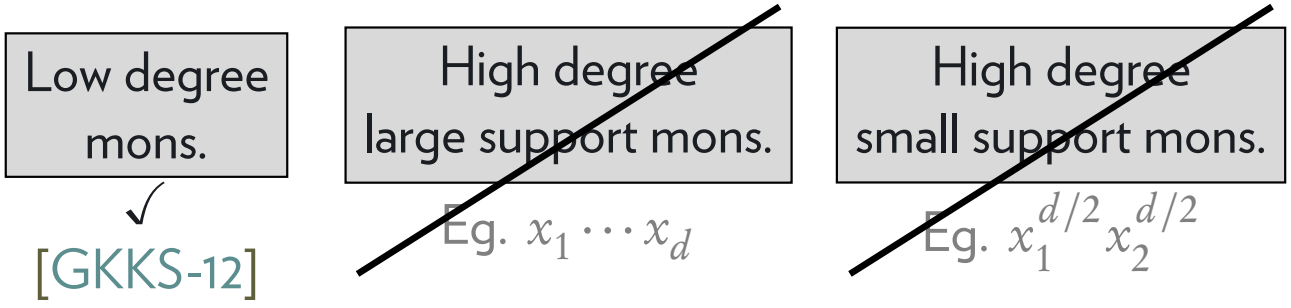

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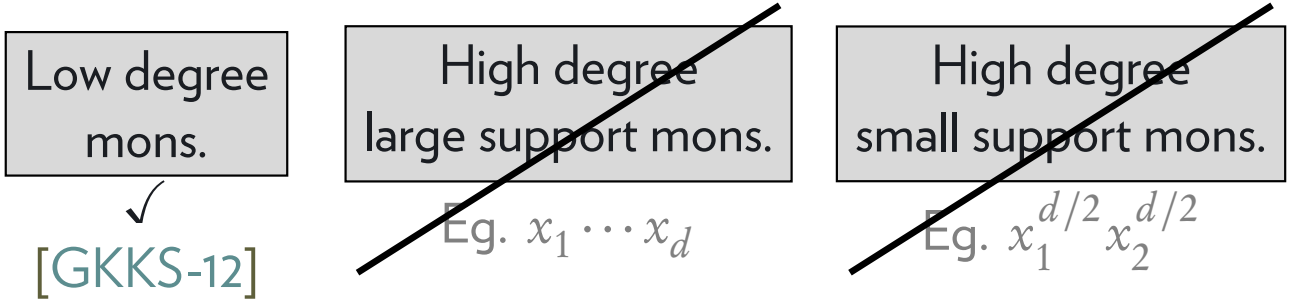

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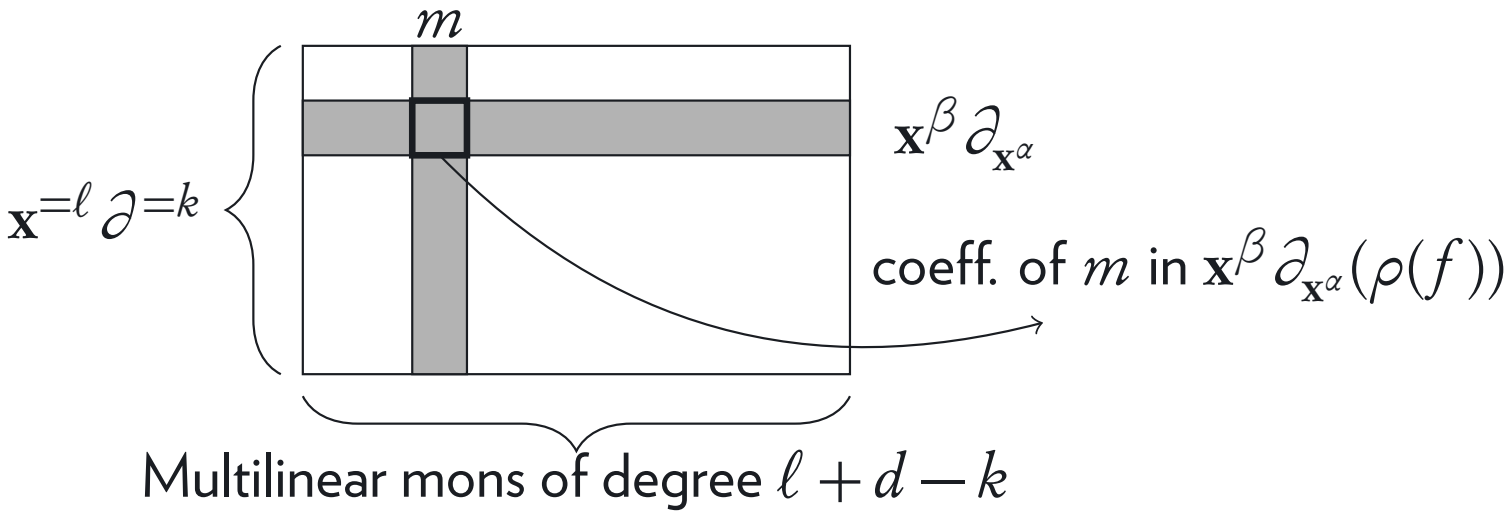

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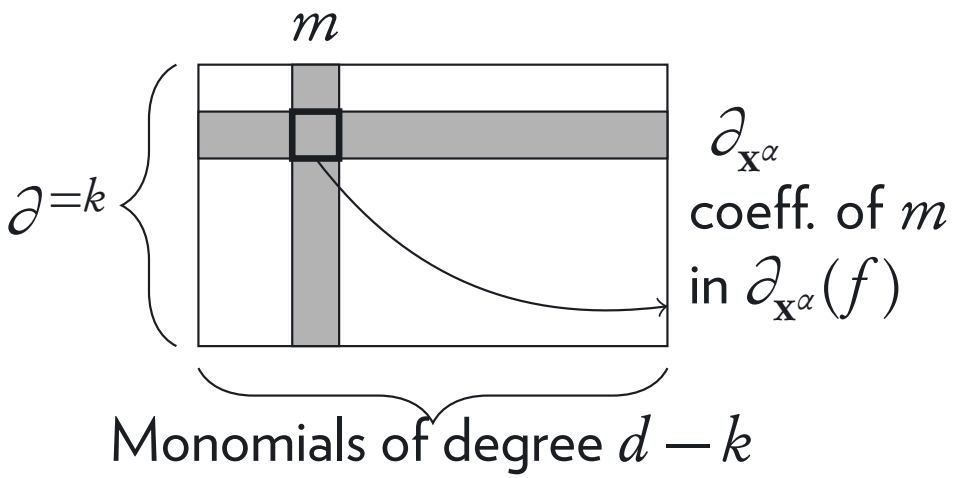

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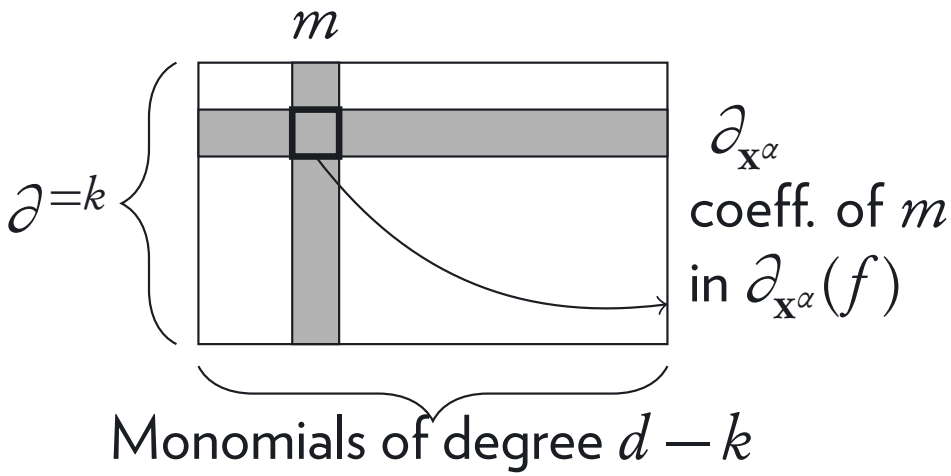

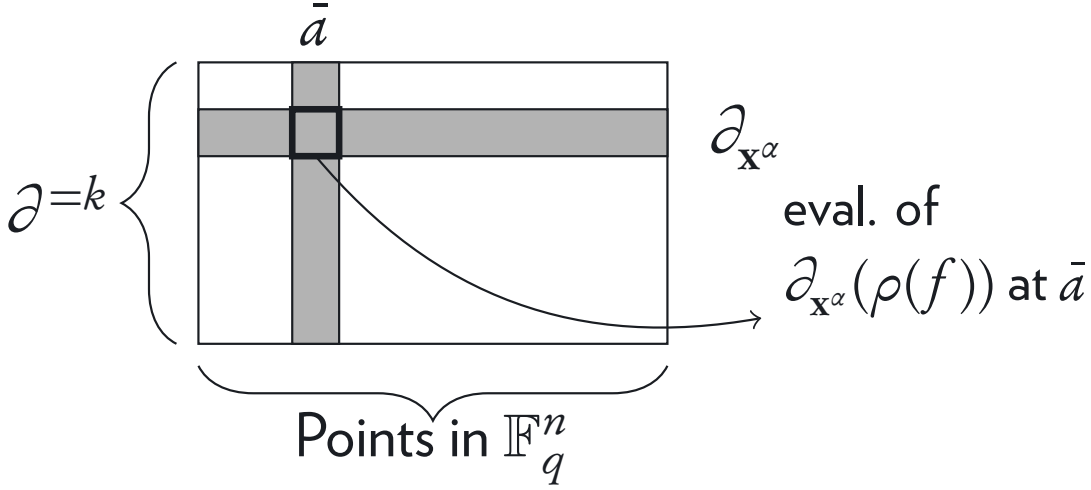

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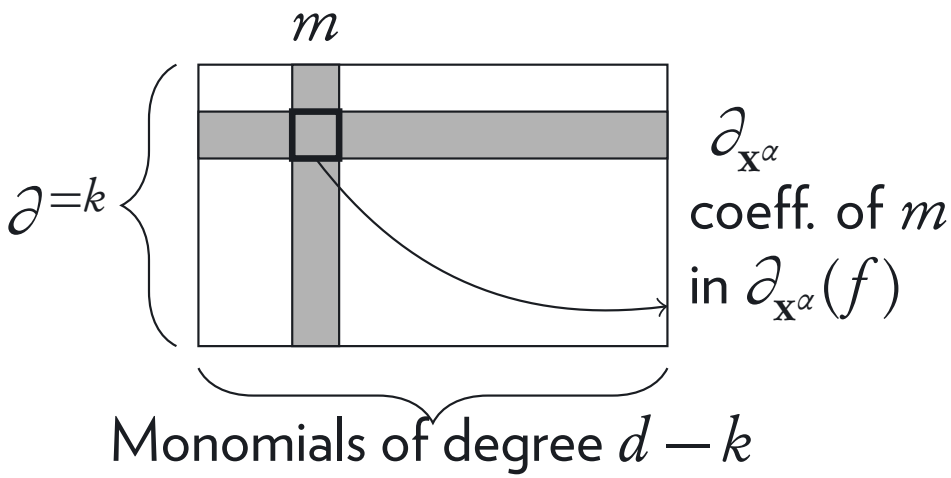

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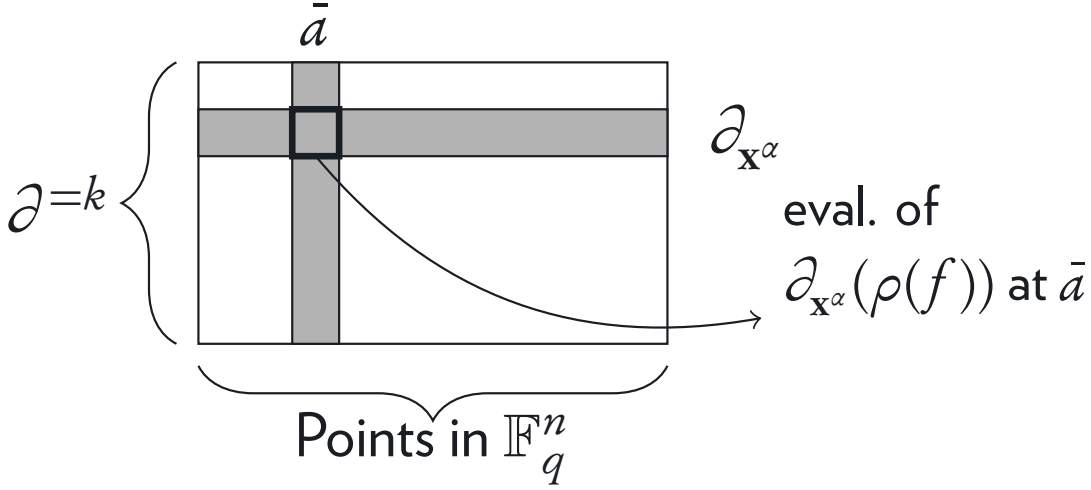

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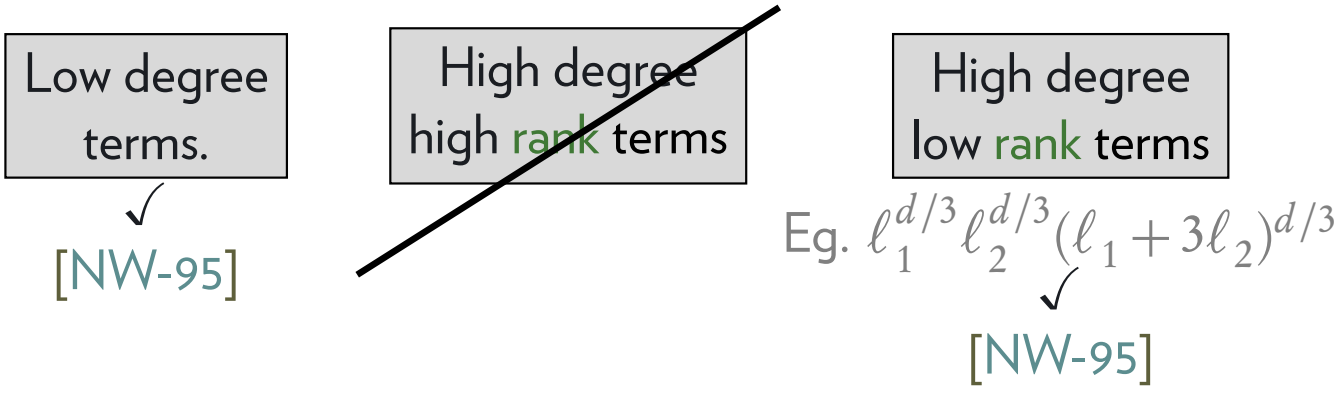

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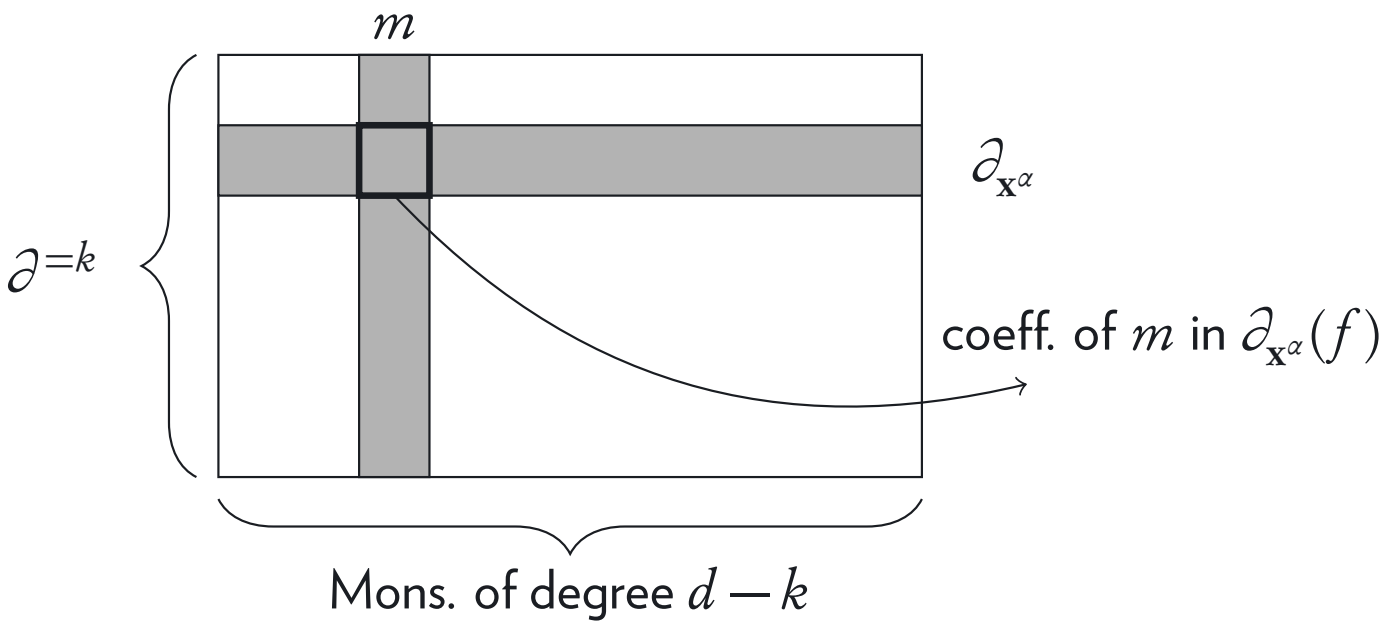

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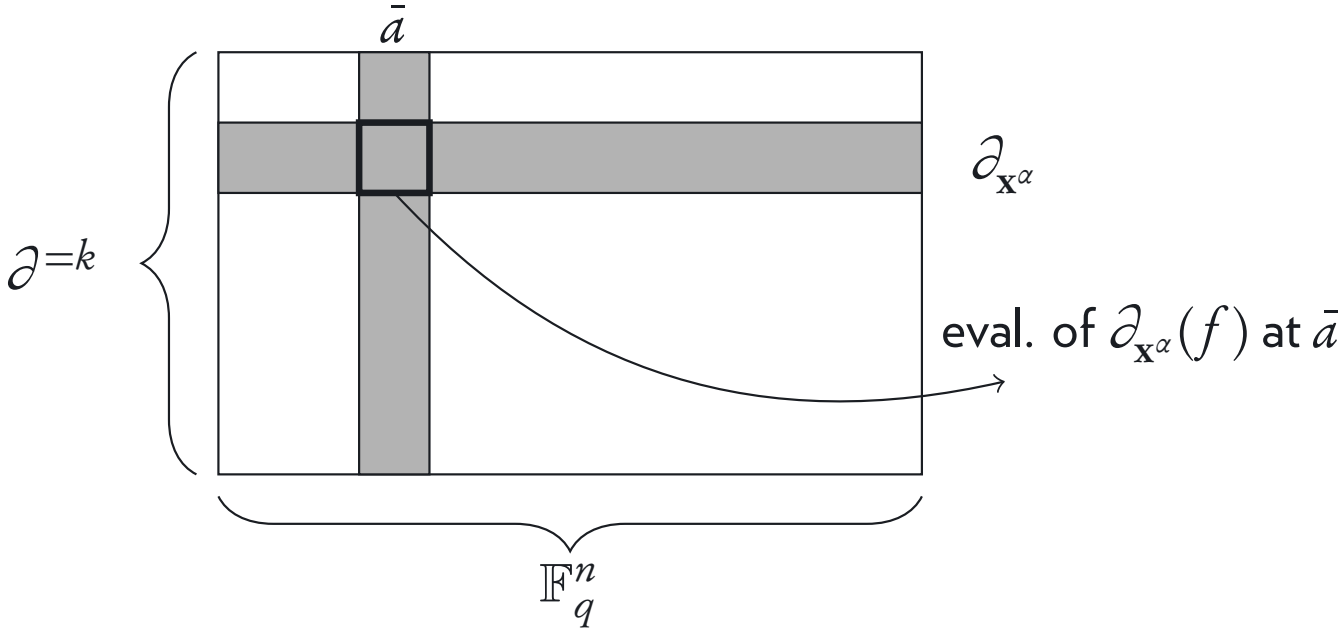

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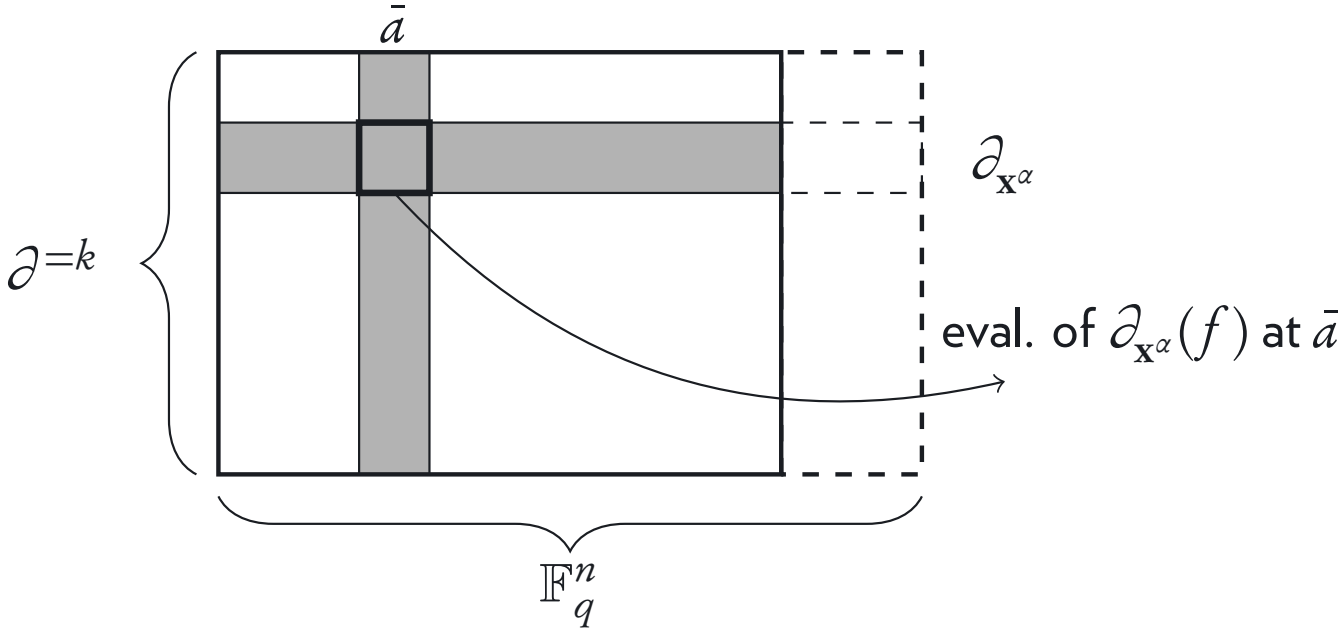

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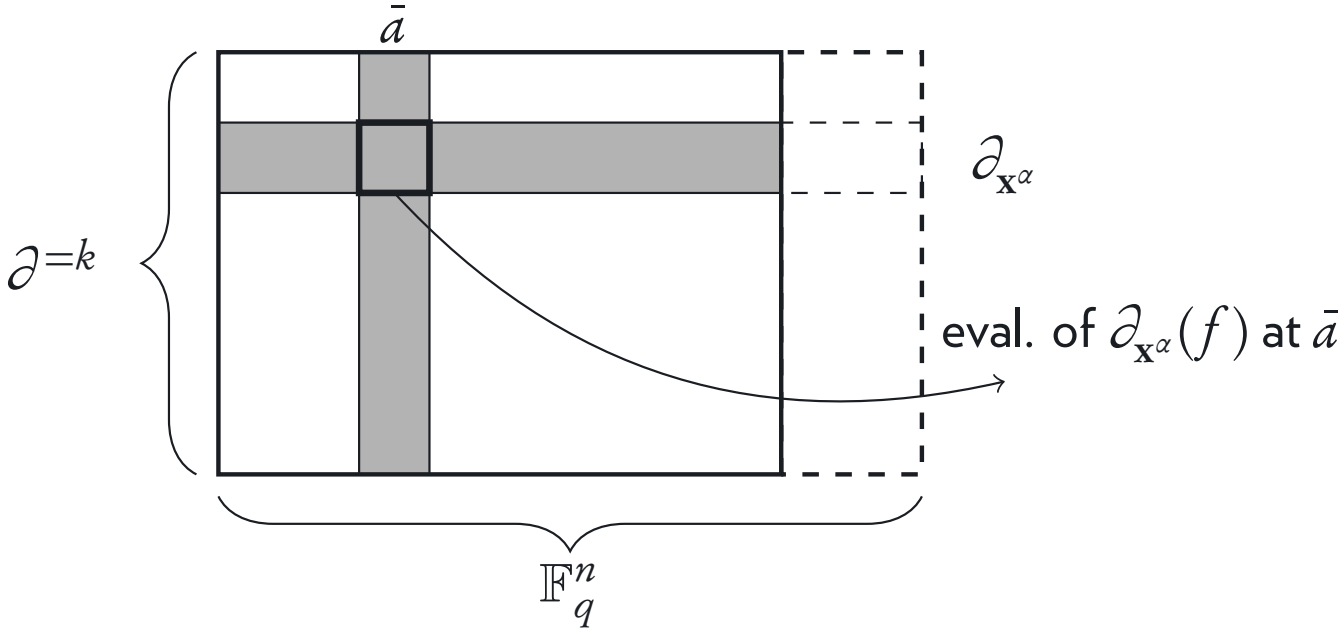

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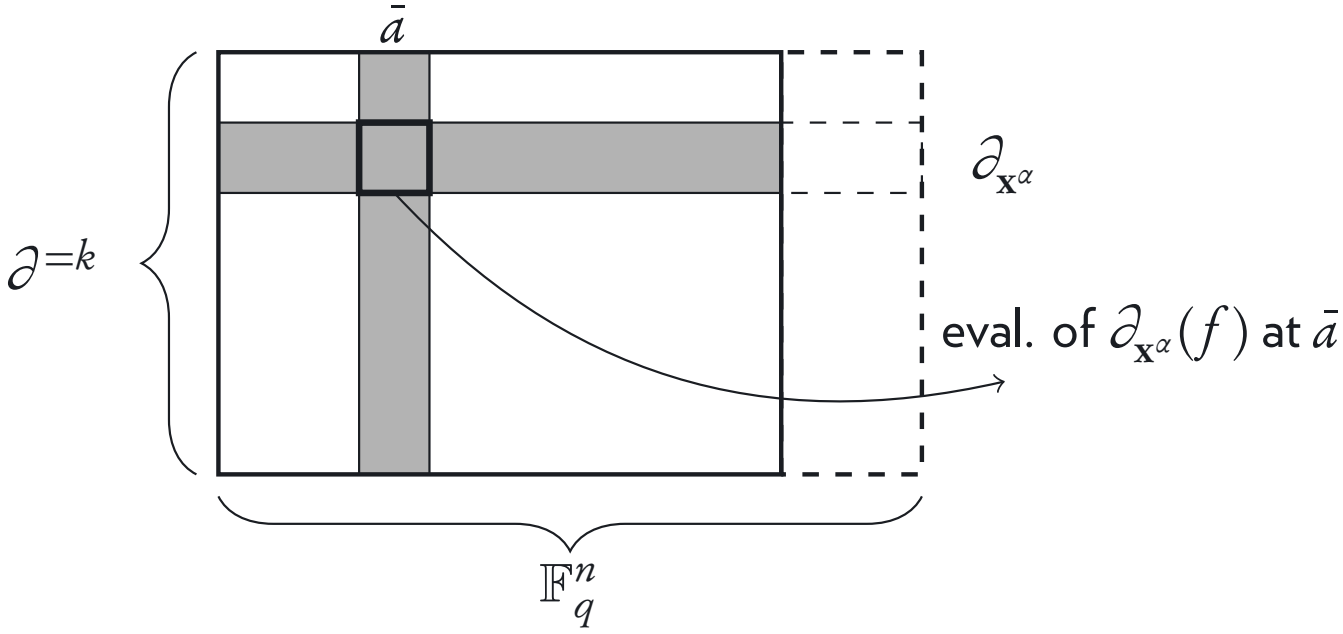

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For Det_n or Perm_n the above matrix remains full rank, as long as we removed only few columns.

Lifting to depth five

$\Sigma\Pi\Sigma\Pi\Sigma$

Types of products of linear polynomials:

Low degree products.	High degree products.

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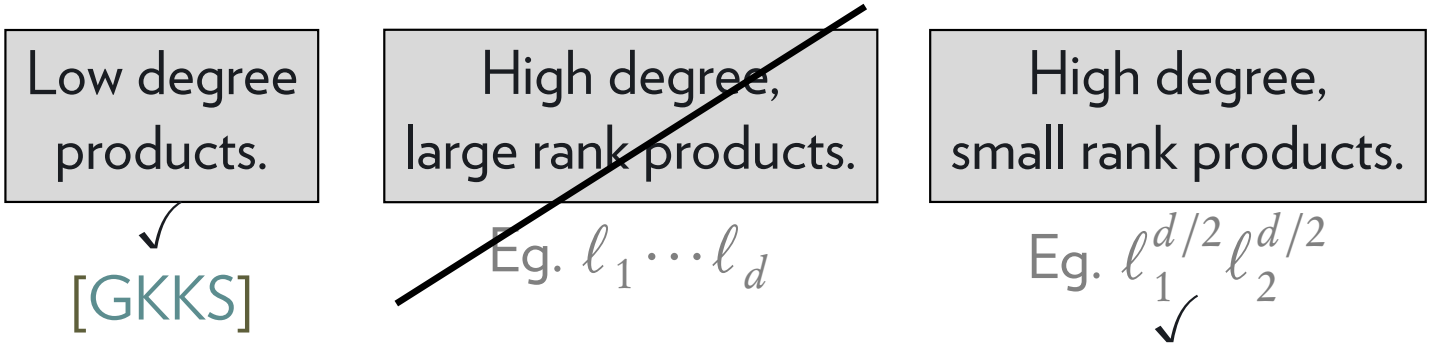

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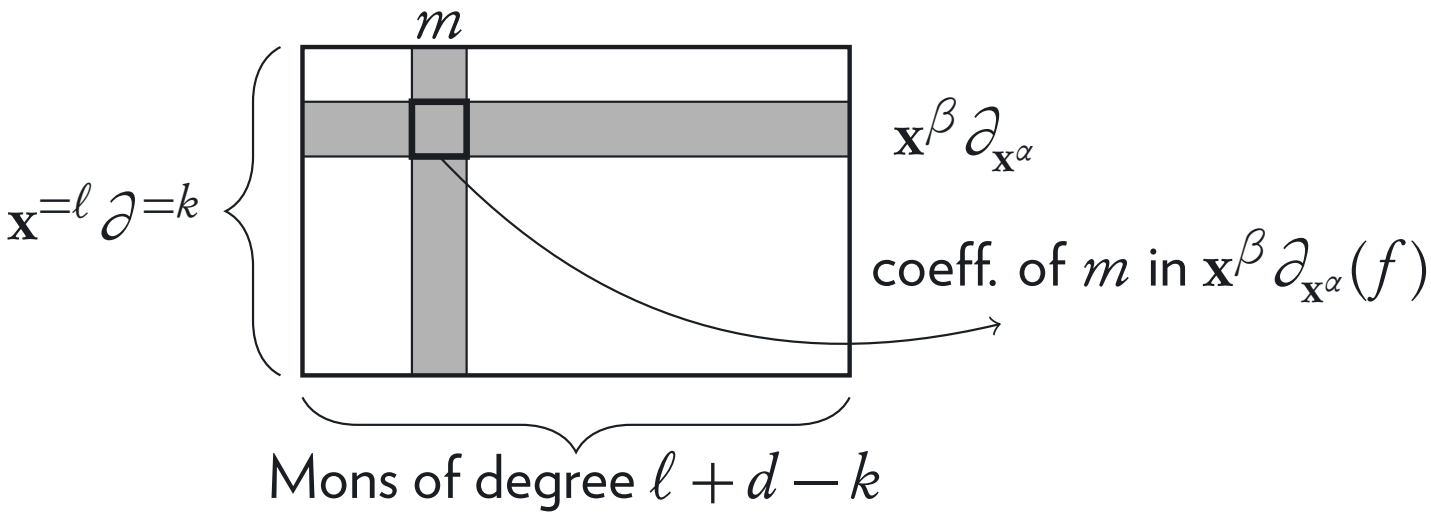

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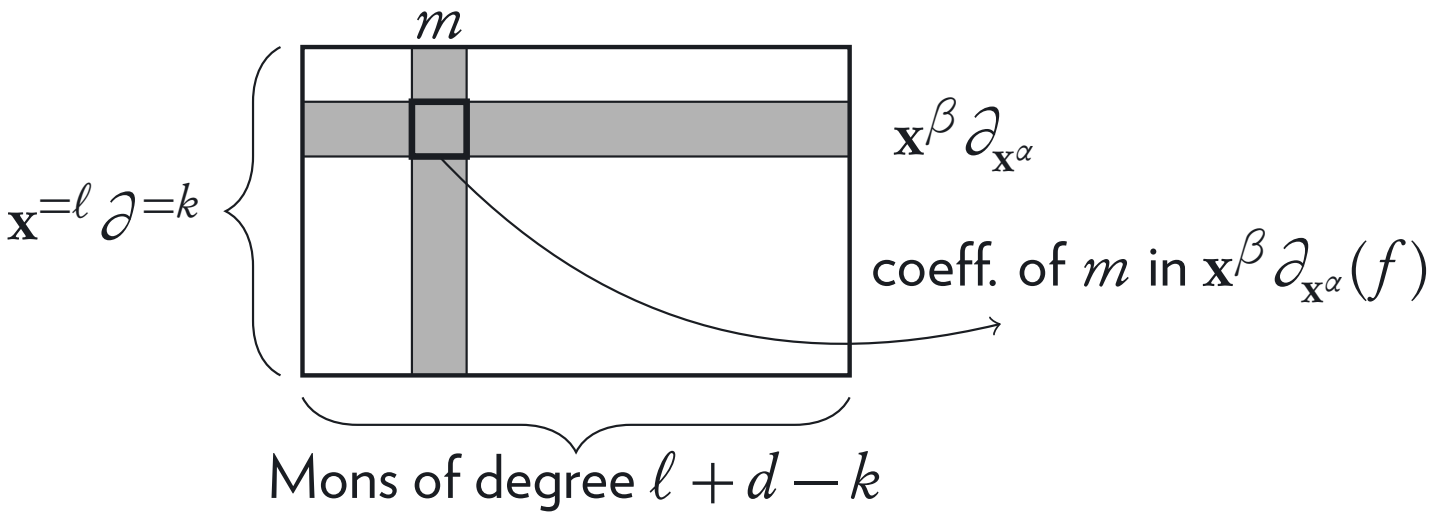

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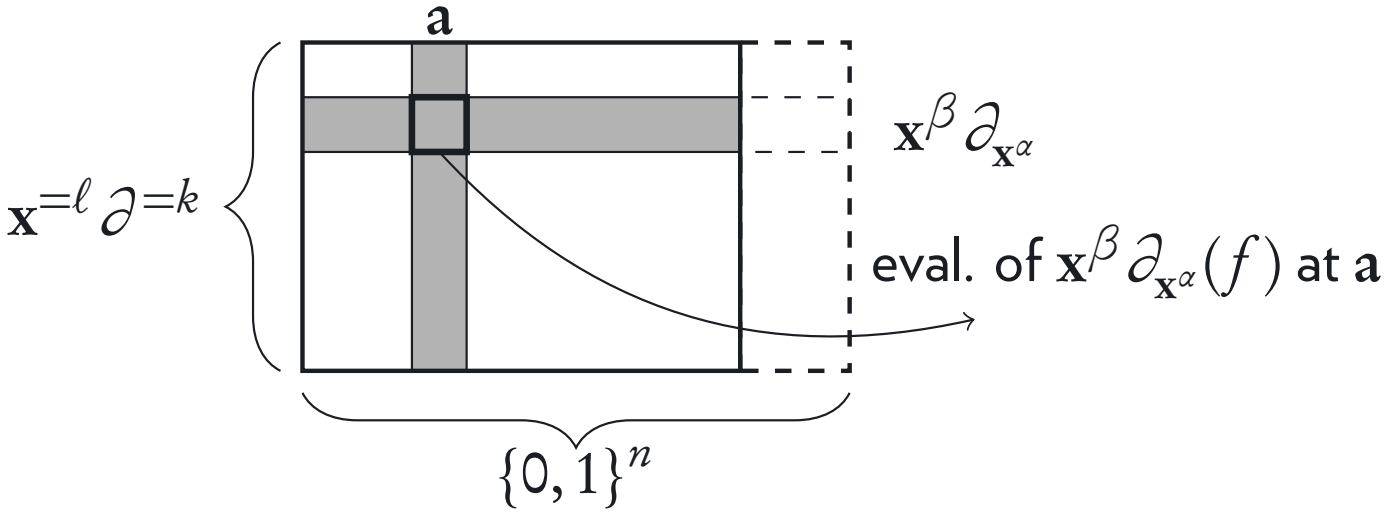

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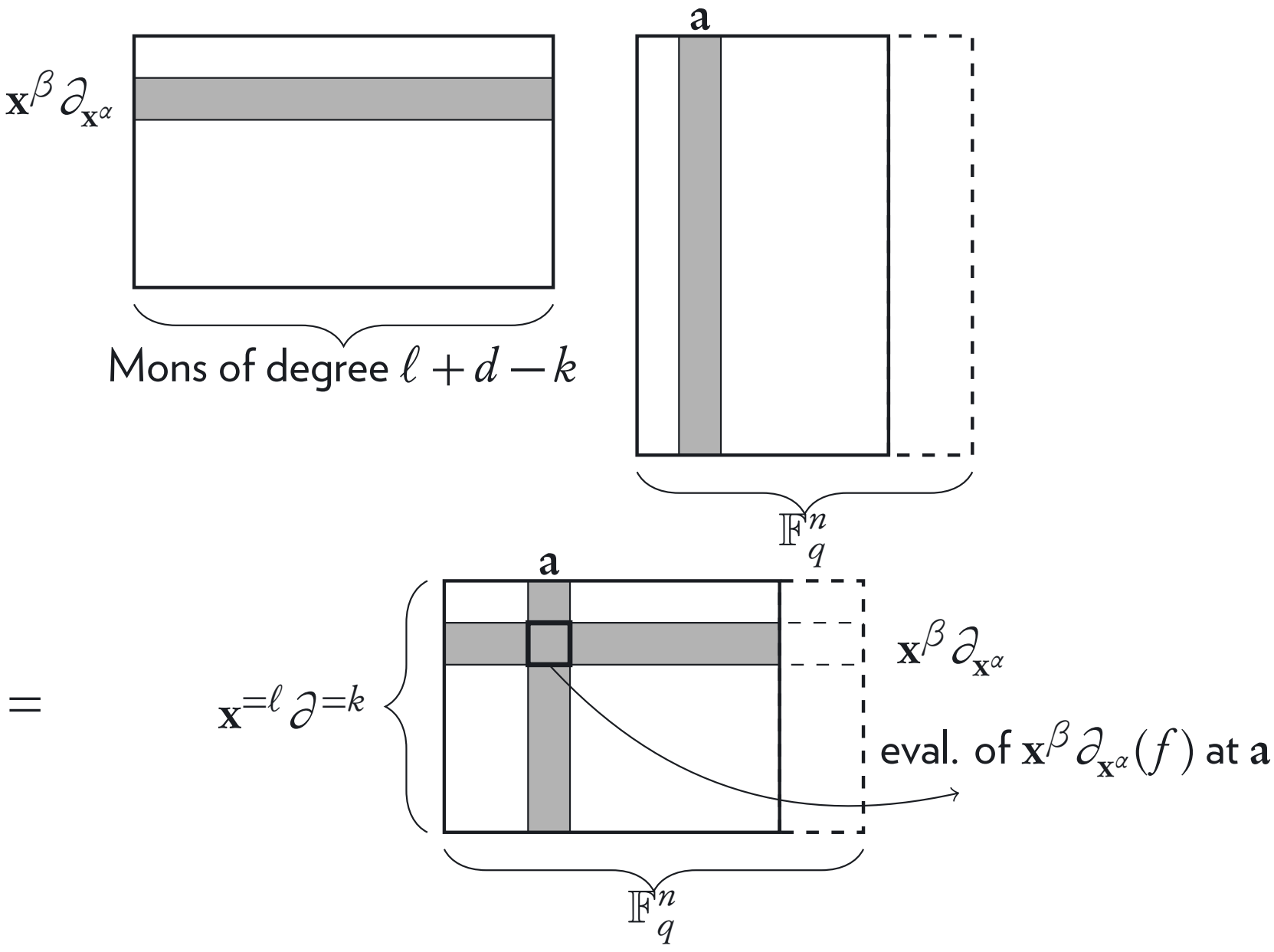

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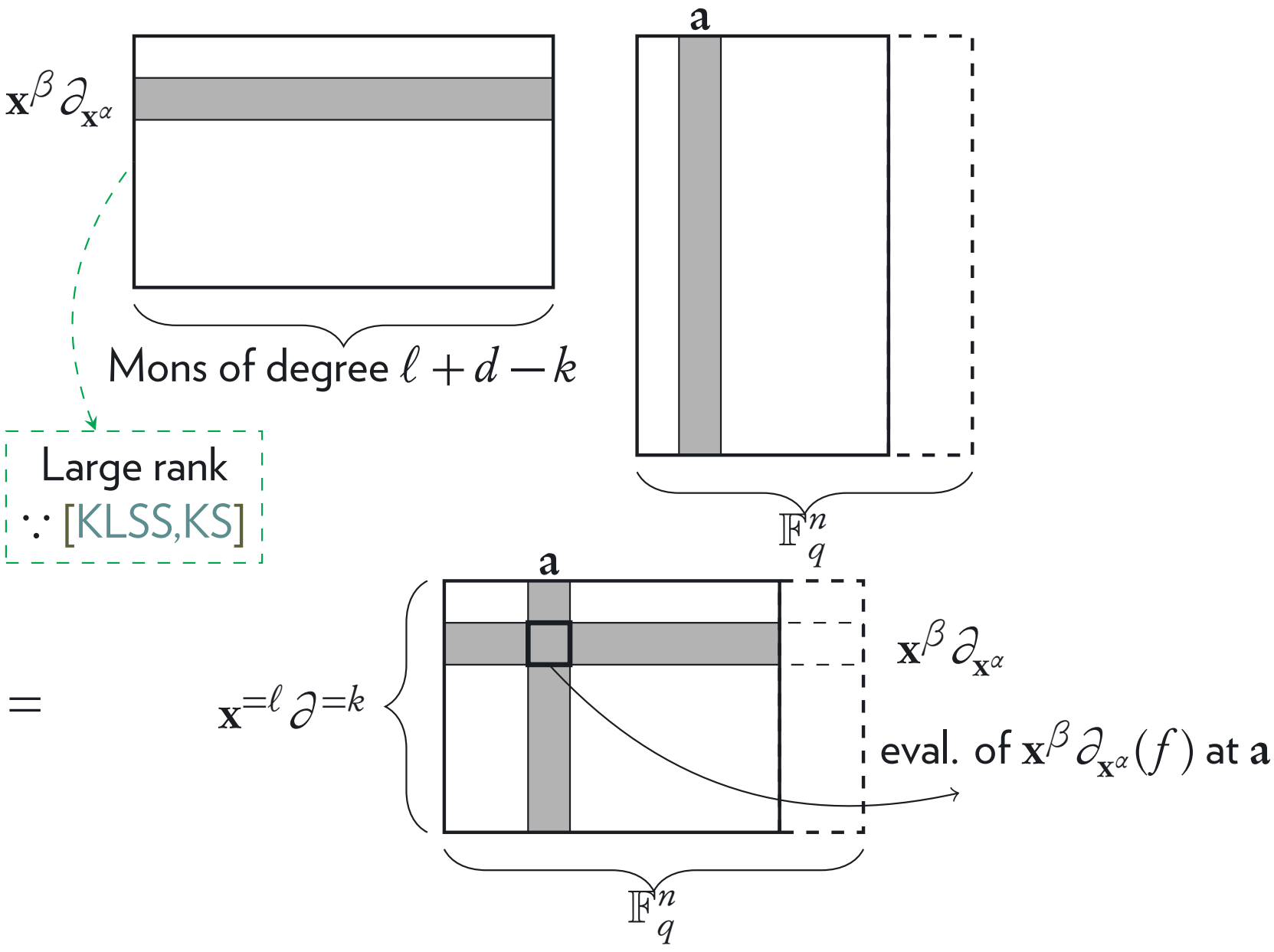

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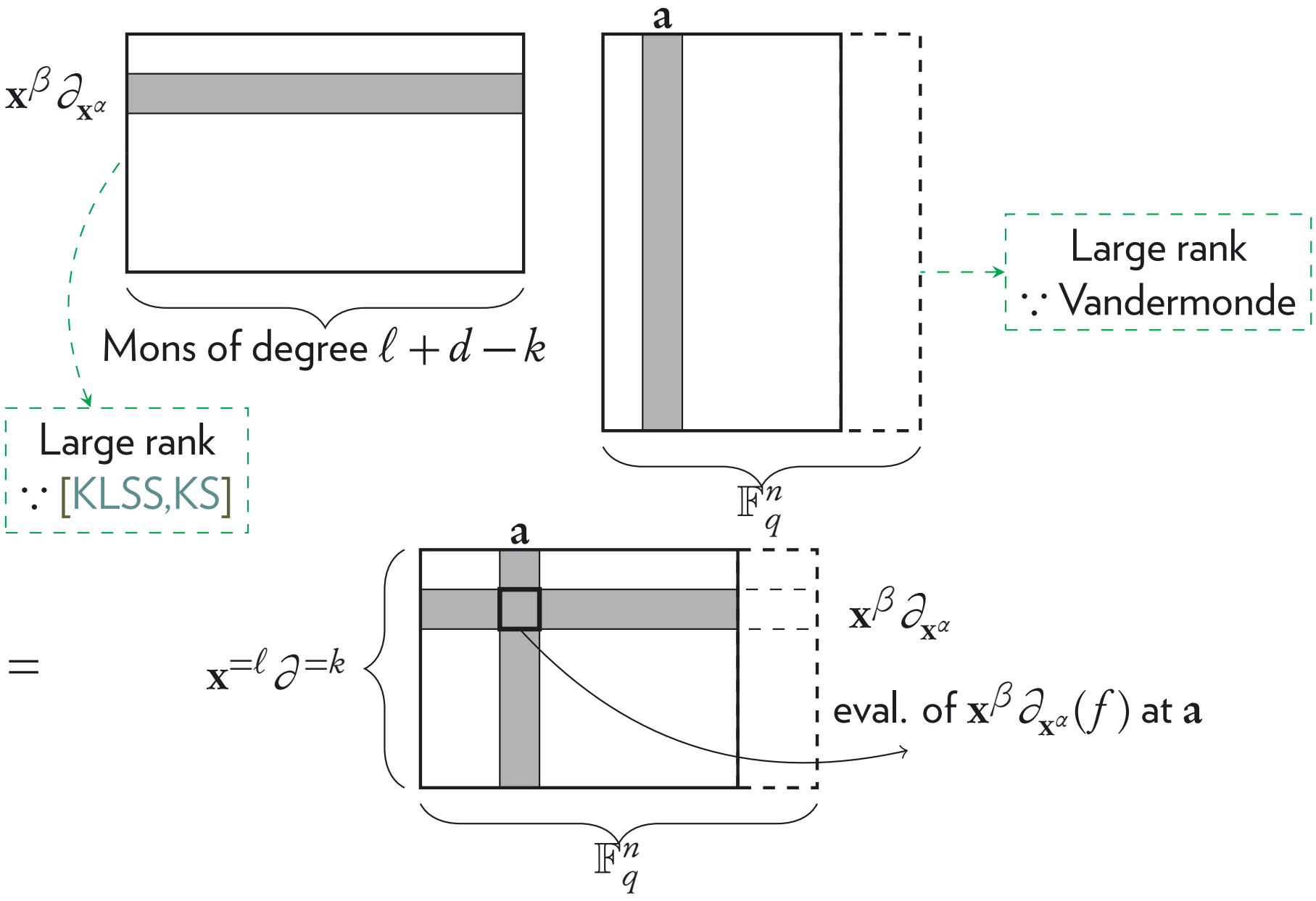

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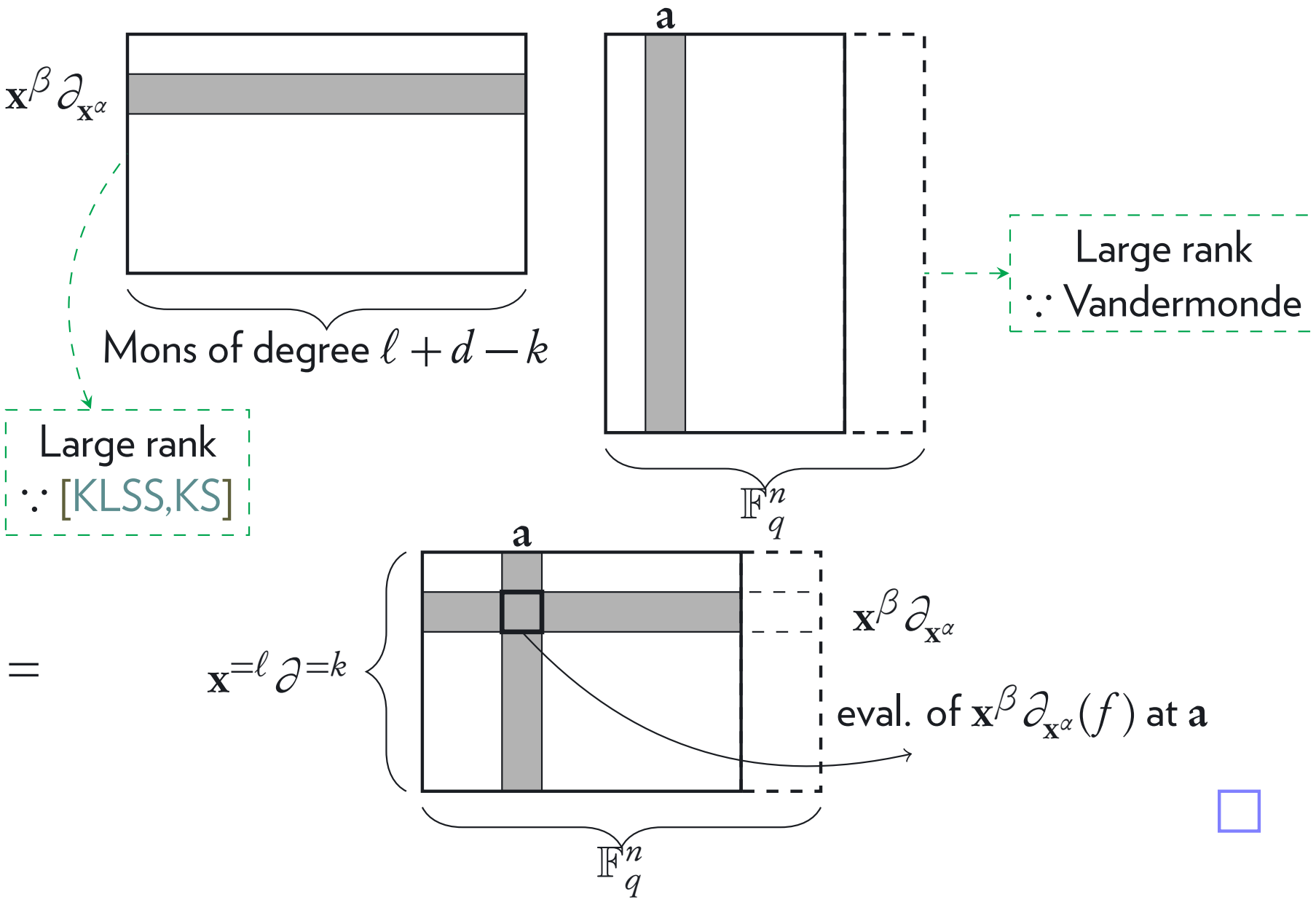

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(Barely manages to work for a specific explicit polynomial. Phew!) □

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- Other fields?

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