Trading Information Complexity for Error

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Main results

- How much information one can save by allowing an error ϵ.

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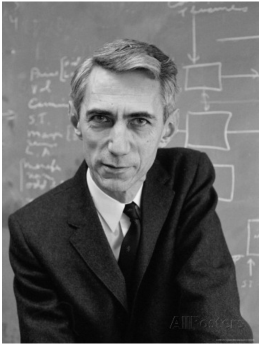
- Showed a separation between two concepts in information complexity.
- Determined communication complexity of computing disjointness function with error ϵ.

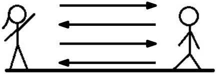
Information complexity

Extension of Shannon's information theory towards studying communication complexity.

Shannon (1916-2001)



Communication complexity



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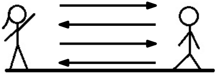

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- They want to compute $f(X,Y) : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$ collaboratively using a protocol.

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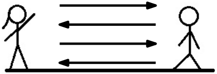

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- A protocol π is an algorithm that defines what Alice and Bob do, in order to compute $f(X,Y)$.

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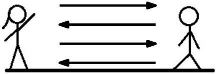

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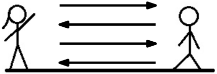
- A protocol π is an algorithm that defines what Alice and Bob do, in order to compute $f(X,Y)$.
- $CC(\pi) :=$ how many bits of communication are sent in π ?

Information complexity



- Same setting;

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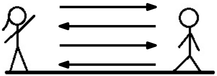

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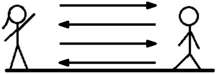

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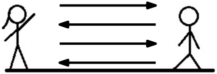

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Information cost of a protocol π:
$\mathrm{IC}_\mu(\pi) =$ information about Y that Alice learns $+$
information about X that Bob learns.

Information cost: Example 1

AND: $\{0,1\} \times \{0,1\} \to \{0,1\}$

Product distribution μ: $\Pr[X=1] = \frac{1}{2}, \quad \Pr[Y=1] = \frac{1}{2}$.

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- Alice learns Bob's input: learned information $= H(Y) = 1$;

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- Bob learns Alice's input: learned information $= H(X) = 1$;
- Information cost of π:

$$\mathrm{IC}_\mu(\pi) = 1 + 1 = 2 = CC(\pi).$$

Information cost: Example 2

AND: $\{0,1\} \times \{0,1\} \to \{0,1\}$

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Information cost: Example 2

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Better protocol τ

- Alice sends her input X to Bob;
- Bob computes and outputs $\mathrm{AND}(X, Y)$.

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Note: $CC(\tau) = 2 = CC(\pi)$.

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Why is the protocol τ better than π?

Information cost: Example 2 - continued

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- Alice learns information $= \frac{1}{2}H(Y) = 0.5$.

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- Information cost of the protocol τ:

$$\mathrm{IC}_\mu(\tau) = 1 + 0.5 = 1.5 < 2 = \mathrm{IC}_\mu(\pi) \Longrightarrow \tau \text{ is better!}.$$

Information complexity of a function f

Definition

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Information complexity of AND

Theorem (BGPW'13)

Let μ be the uniform distribution, then

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$$\max_\mu \mathrm{IC}_\mu(\mathrm{AND}, 0) \approx 1.49...$$

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$\mathrm{IC}_\mu(\pi) = 1 + 1 = 2$.

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Great Idea: ALLOW ERROR!

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Q: How much information can one save?

$$\mathrm{IC}_\mu(\mathrm{XOR}, 0) - \mathrm{IC}_\mu(\mathrm{XOR}, \epsilon) = ?$$

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Information cost $\mathrm{IC}_\mu(\pi_\epsilon) = ?$

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- Bayesian:

$$\Pr[X = 1 | X' = 1] = \frac{\Pr[X' = 1 | X = 1] \Pr[X = 1]}{\Pr[X' = 1]} = 1 - \epsilon.$$

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- Bob learns less information about Alice's input. Indeed,

$$\mathrm{IC}_\mu(\pi_\epsilon) = 1 + (1 - H(1 - \epsilon)) = 2 - H(\epsilon).$$

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Hence the information saved by allowing error ϵ:

$$\mathrm{IC}_\mu(\mathrm{XOR}, 0) - \mathrm{IC}_\mu(\mathrm{XOR}, \epsilon) \geq H(\epsilon).$$

Analysis of π_ϵ

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Hence the information saved by allowing error ϵ:

$$\mathrm{IC}_\mu(\mathrm{XOR},0)-\mathrm{IC}_\mu(\mathrm{XOR},\epsilon)\ge H(\epsilon). \quad \text{True for all functions } f?$$

Main results

Theorem (Dagan-Filmus-Hatami-L'16)

For all functions f, for all distributions μ such that $\mathrm{IC}_\mu(f,0) > 0$,

$$\Omega(H(\epsilon)) \le \mathrm{IC}_\mu(f,0) - \mathrm{IC}_\mu(f,\epsilon) \le O(H(\sqrt{\epsilon})).$$

Theorem (Dagan-Filmus-Hatami-L'16)

For all μ such that $\mathrm{IC}_\mu(\mathrm{AND},0) > 0$,

$$\mathrm{IC}_\mu(\mathrm{AND},0) - \mathrm{IC}_\mu(\mathrm{AND},\epsilon) = \Theta(H(\epsilon)).$$

Proof idea of lower bound

Theorem (Dagan-Filmus-Hatami-L'16)

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Proof idea - converting a zero-error protocol π to ϵ error π_ϵ

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- Alice smartly flips her input X with probability ϵ to X';

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Proof idea - converting a zero-error protocol π to ϵ error π_ϵ

- Alice smartly flips her input X with probability ϵ to X';
- They compute f using π with (X', Y).
- Show

$$\mathrm{IC}_\mu(\pi) - \mathrm{IC}_\mu(\pi_\epsilon) \geq \Omega(H(\epsilon)).$$

Tight bound for AND

Theorem (Dagan-Filmus-Hatami-L'16)

$$\mathrm{IC}_\mu(\mathrm{AND}, 0) - \mathrm{IC}_\mu(\mathrm{AND}, \epsilon) \leq O(H(\epsilon)).$$

Tight bound for AND

Theorem (Dagan-Filmus-Hatami-L'16)

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Some new ideas

- study the stability of protocols;
  may be useful to study other functions.
- parametrization of all distributions by product distributions.
  applicable to all functions!

Worst case error v.s. distributional error

Recall

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Obviously, $\mathrm{IC}_\mu^D(f, \epsilon) \leq \mathrm{IC}_\mu(f, \epsilon)$, and the inequality can be strict!

Prior-free information complexity

Definition [Braverman'12]

$$\mathrm{IC}(f,\epsilon) := \max_{\mu} \mathrm{IC}_{\mu}(f,\epsilon), \qquad \mathrm{IC}^{D}(f,\epsilon) := \max_{\mu} \mathrm{IC}^{D}_{\mu}(f,\epsilon).$$

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If $\epsilon = 0$, i.e., no error,

$$\mathrm{IC}(f,0) = \mathrm{IC}^{D}(f,0).$$

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Q: $\mathrm{IC}^{D}(f,\epsilon) = \mathrm{IC}(f,\epsilon)$?

Main results - continued

Theorem (Dagan-Filmus-Hatami-L'16)

For n sufficiently large,

$$\mathrm{IC}^{D}(\mathrm{DISJ}_n, \epsilon) < \mathrm{IC}(\mathrm{DISJ}_n, \epsilon).$$

Implication on communication complexity

Theorem (BGPW'13)

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Theorem (Dagan-Filmus-Hatami-L'16)

$$CC(\mathrm{DISJ}_n, \epsilon) \approx (0.4827 - \Theta(H(\epsilon)))n.$$

Proof: follows from results on $\mathrm{IC}(\mathrm{AND}, \epsilon)$. Conjectured by [BGPW13].

Summary and Open Problems

Theorem (Dagan-Filmus-Hatami-L'16)

- $\Omega(H(\epsilon)) \leq \mathrm{IC}_\mu(f, 0) - \mathrm{IC}_\mu(f, \epsilon) \leq O(H(\sqrt{\epsilon}))$;
- $\mathrm{IC}^D(f, \epsilon) \neq \mathrm{IC}(f, \epsilon)$, *example:* DISJ_n;
- $CC(\mathrm{DISJ}_n, \epsilon) \approx (0.4827 - \Theta(H(\epsilon)))n$.

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Open Problem

- $\mathrm{IC}_\mu(f,0) - \mathrm{IC}_\mu(f,\epsilon) = \Theta(H(\epsilon))$?
- $\lim_{n\to\infty} \frac{\mathrm{IC}(\mathrm{DISJ}_n,\epsilon)}{n} = \max_{\mu:\mu(11)=0} \mathrm{IC}_\mu(\mathrm{AND},\epsilon,1\to 0)$?