

# On algebraic branching programs of small width

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## Small width algebraic branching programs: surprisingly powerful

1. Width-2 algebraic branching programs with approximation are as powerful as formulas
2. Width-1 algebraic branching programs with nondeterminism are as powerful as circuits

## 1. Definitions

- Algebraic branching programs
- Formulas
- Complexity classes  $\mathbf{VP}_k$  and  $\mathbf{VP}_e$
- Approximation classes  $\overline{\mathbf{VP}}_k$  and  $\overline{\mathbf{VP}}_e$

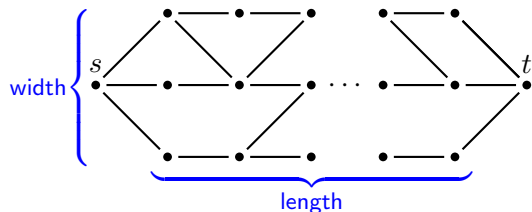
## 2. Historical context

## 3. Statement of main result

## 4. Proof sketch

## 5. Statement of nondeterminism result

# Algebraic branching program (ABP) definition



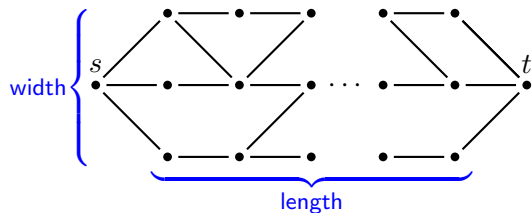
edge labels are  
affine linear forms:

$$\alpha_0 + \alpha_1 x_1 + \cdots + \alpha_n x_n$$

$(\alpha_i \in \mathbb{C})$

$$f(x_1, \dots, x_n) = \sum_{\substack{s-t \text{ paths} \\ \text{in graph}}} \text{product of edge labels on path}$$

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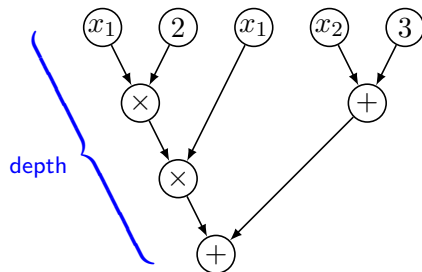
Example

$$x^2 + y^2 + z^2 = \sum_{s-t \text{ path products}} \begin{array}{c} \begin{array}{ccccc} & & x & & x & & \\ & & \bullet & \text{---} & \bullet & \text{---} & \bullet \\ & & & & & & \\ s & & \bullet & \text{---} & \bullet & \text{---} & \bullet & \text{---} & \bullet & & t \\ & & & & & & \\ & & z & & z & & \end{array} \end{array}$$

Complexity

$L_k(f)$  = minimum length of any width- $k$  ABP computing  $f$

# Formula definition



size = number of nodes

$f(x_1, \dots, x_n)$  = evaluation of tree

Complexity

$L_e(f)$  = minimum size of any formula computing  $f$

leaves

variables  $x_i$

constants  $\alpha_i \in \mathbb{C}$

nodes

$+$ ,  $\times$

fan-in 2

fan-out 1

# Classes $\mathbf{VP}_k$ and $\mathbf{VP}_e$ definition

- Recall:
- $L_k =$  width- $k$  ABP length
  - $L_e =$  formula size

**family:** sequence  $(f_n)_{n \in \mathbb{N}}$  of polynomials  $f_n(x_1, \dots, x_{\text{poly}(n)})$

$\mathbf{VP}_k := \{ \text{families } (f_n)_{n \in \mathbb{N}} \text{ with } L_k(f_n) = \text{poly}(n) \}$   $k \in \mathbb{N}$

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**Ben-Or and Cleve (1988)** inspired by Barrington's theorem (1986)

$\mathbf{VP}_3 = \mathbf{VP}_4 = \dots = \mathbf{VP}_e$

In particular: width-3 ABPs can compute any polynomial

**Allender and Wang (2011)**

Strict inclusion:  $\mathbf{VP}_2 \subsetneq \mathbf{VP}_3$

No width-2 ABP computes  $x_1x_2 + \dots + x_{15}x_{16}$



# Approximation

$\sum_{s-t \text{ path products}}$

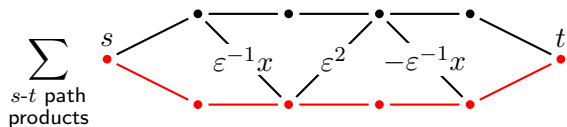
$= 1 + 1 + \varepsilon^{-1}x - \varepsilon^{-1}x + \varepsilon x - \varepsilon x - x^2 + \varepsilon^2$

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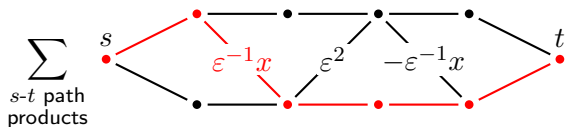
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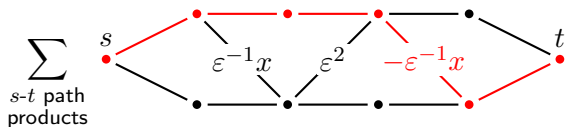
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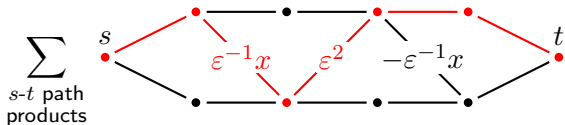
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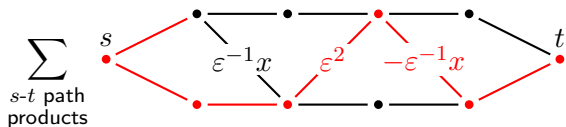
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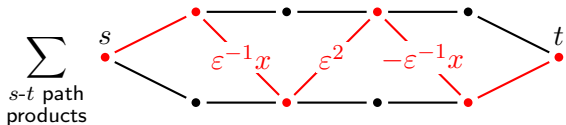
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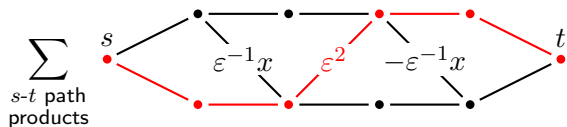
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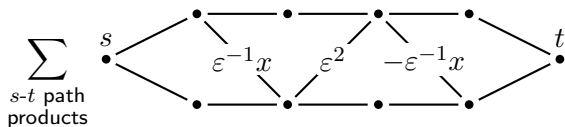


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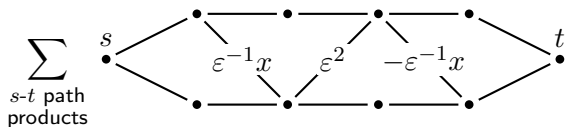
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$$= 1 + 1 + \epsilon^{-1}x - \epsilon^{-1}x + \epsilon x - \epsilon x - x^2 + \epsilon^2$$

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# Approximation



$$= 1 + 1 + \varepsilon^{-1}x - \varepsilon^{-1}x + \varepsilon x - \varepsilon x - x^2 + \varepsilon^2$$

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- $2 - x^2 + \varepsilon^2 \xrightarrow{\varepsilon \rightarrow 0} 2 - x^2$
- $L_2(2 - x^2 + \varepsilon^2) \leq 4 \quad (\varepsilon > 0)$

We say " $\overline{L}_2(2 - x^2) \leq 4$ "

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Border complexity    cp. border rank (Bini et al., Strassen)

$V = \mathbb{C}[x_1, \dots, x_n]_{\leq d}$  degree  $\leq d$  polyn. endowed with **Euclidean norm**

$\overline{L}(f) :=$  smallest  $r$  for which there exist  $(g_\varepsilon)_{\varepsilon \in \mathbb{R}_{>0}} \subseteq V$  and

- $\lim_{\varepsilon \rightarrow 0} g_\varepsilon = f$
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$\overline{\mathbf{VP}}_k = \{ \text{families } (f_n)_{n \in \mathbb{N}} \text{ with } \overline{L}_k(f_n) = \text{poly}(n) \}$   $k \in \mathbb{N}$

$\overline{\mathbf{VP}}_e = \{ \text{families } (f_n)_{n \in \mathbb{N}} \text{ with } \overline{L}_e(f_n) = \text{poly}(n) \}$

Clearly  $\overline{L}(f) \leq L(f)$ . Therefore  $\mathbf{VP}_k \subseteq \overline{\mathbf{VP}}_k$ ,  $\mathbf{VP}_e \subseteq \overline{\mathbf{VP}}_e$ , etc

## More historical context

Valiant (1979)

$$\mathbf{VP}_e \subseteq \mathbf{VP}_s \subseteq \mathbf{VP} \subseteq \mathbf{VNP}$$

*Valiant's conjectures*

$$\mathbf{VP}_e, \mathbf{VP}_s, \mathbf{VP} \stackrel{?}{\not\subseteq} \mathbf{VNP}$$

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Strassen, Mulmuley-Sohoni (GCT),  
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Extended conjectures  $\overline{\mathbf{VP}}_s, \overline{\mathbf{VP}} \stackrel{?}{\not\subseteq} \mathbf{VNP}$

Proving e.g.  $\mathbf{VP}_e \not\subseteq \mathbf{VNP}$  using any geometric technique  
(e.g. shifted partial derivatives or geometric complexity theory)  
automatically implies  $\overline{\mathbf{VP}}_e \not\subseteq \mathbf{VNP}$ .

We study

$\overline{\mathbf{VP}}_e$

Recent work on closures of classes:

Forbes (2016), Grochow-Mulmuley-Qiao (2016)

# Statement of main result

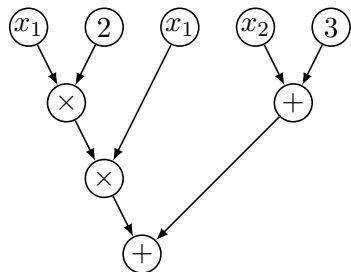
Main theorem:  $\overline{\mathbf{VP}_2} = \overline{\mathbf{VP}_e}$

$$\begin{array}{ccccc} \overline{\mathbf{VP}_2} & = & \overline{\mathbf{VP}_3} & = & \overline{\mathbf{VP}_e} \\ \cup \neq & & \cup & & \cup \\ \mathbf{VP}_2 & \stackrel{\subset \neq}{\uparrow} & \mathbf{VP}_3 & = & \mathbf{VP}_e \\ & \text{Allender-Wang} & \text{Ben-Or-Cleve} & & \end{array}$$

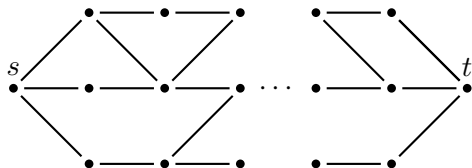
Corollary: strict inclusion  $\mathbf{VP}_2 \subsetneq \overline{\mathbf{VP}_2}$

# Ben-Or and Cleve construction

To prove:  $\mathbf{VP}_e \subseteq \mathbf{VP}_3$



size  $s$  formula  $\rightsquigarrow$



edge labels: affine linear forms

size  $\text{poly}(s)$  width-3 ABP

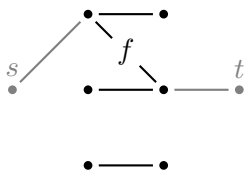
**Brent (1974)** depth reduction:

size  $\text{poly}(s)$

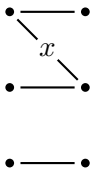
depth  $\mathcal{O}(\log s)$  formula

To prove:  $\mathbf{VP}_e \subseteq \mathbf{VP}_3$

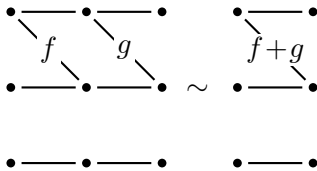
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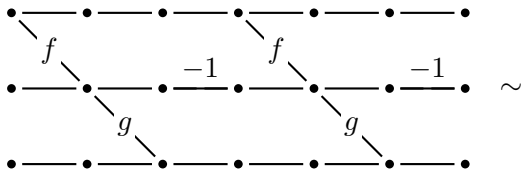
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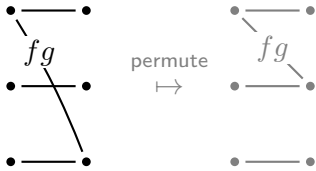
addition



multiplication

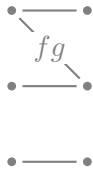


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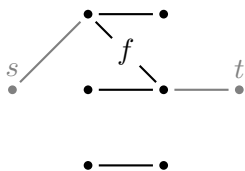
permute

$\mapsto$

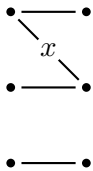


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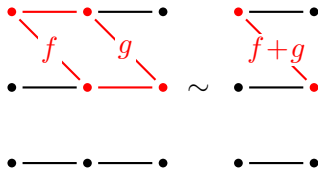
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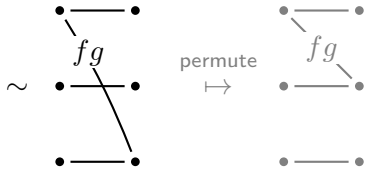
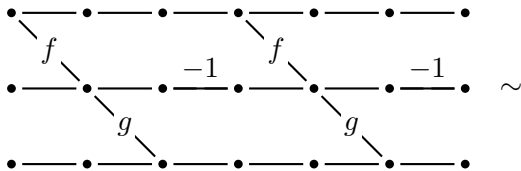
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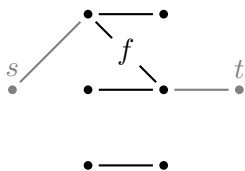


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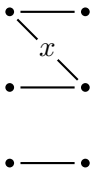


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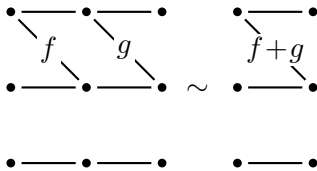
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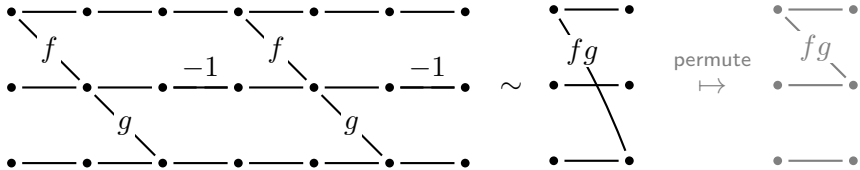
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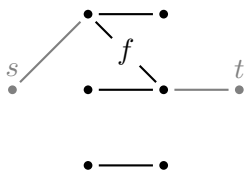


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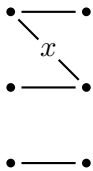


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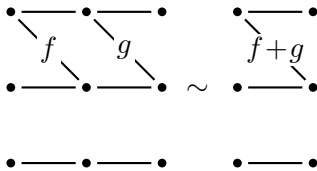
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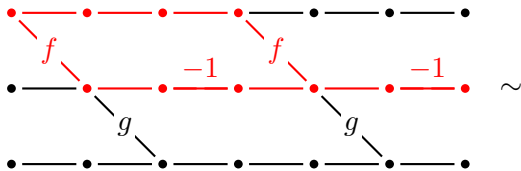
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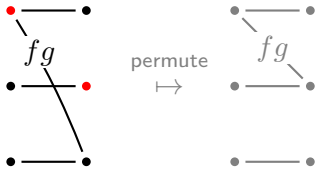
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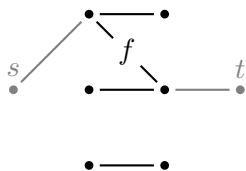


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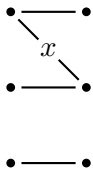


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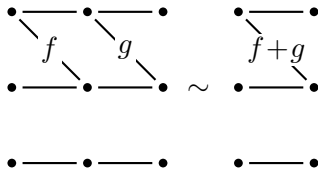
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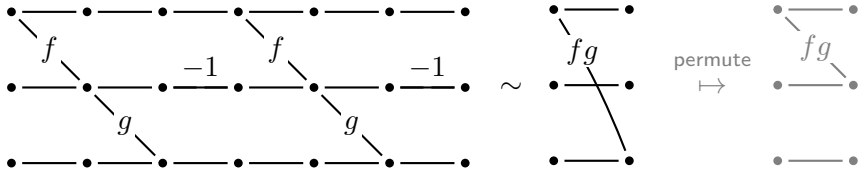
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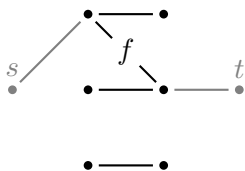
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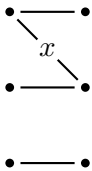


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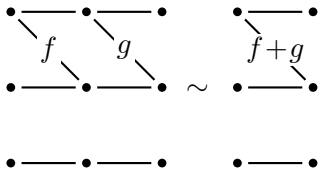
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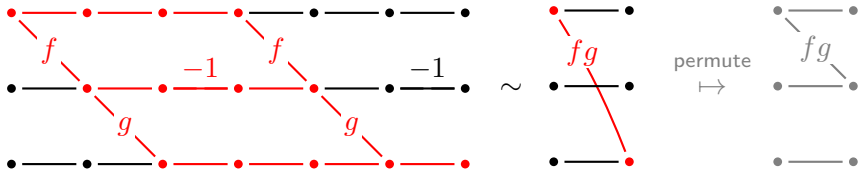
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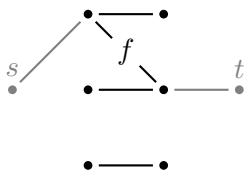


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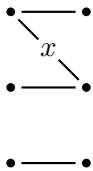


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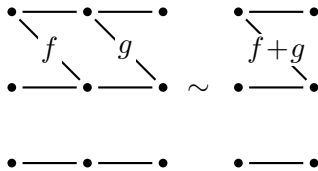
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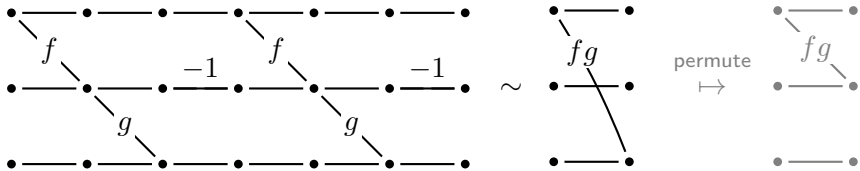
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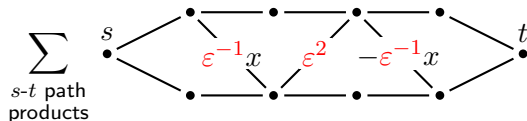
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To prove:  $\mathbf{VP}_e \subseteq \overline{\mathbf{VP}_2}$  (then  $\overline{\mathbf{VP}_e} \subseteq \overline{\mathbf{VP}_2}$  follows)

# Our construction

To prove:  $\mathbf{VP}_\epsilon \subseteq \overline{\mathbf{VP}_2}$  (then  $\overline{\mathbf{VP}_\epsilon} \subseteq \overline{\mathbf{VP}_2}$  follows)

Recall: computational model



$$= 2 + x^2 + \epsilon \xrightarrow{\epsilon \rightarrow 0} 2 + x^2$$

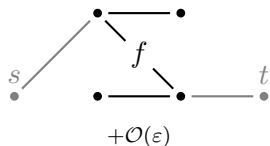
We need

$$= f + \underbrace{\epsilon f_1 + \epsilon^2 f_2 + \dots}_{\mathcal{O}(\epsilon)} \xrightarrow{\epsilon \rightarrow 0} f$$

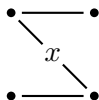
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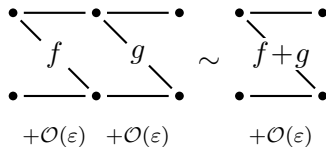
goal



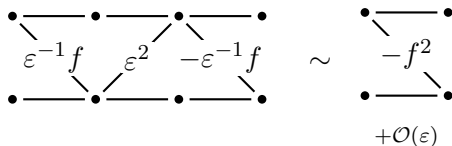
base



addition



squaring (idea)

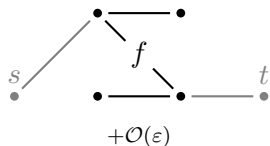


multiplication  $fg = \frac{1}{2}((f+g)^2 - f^2 - g^2)$

# Our construction

To prove:  $\mathbf{VP}_e \subseteq \overline{\mathbf{VP}_2}$

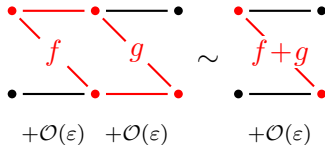
goal



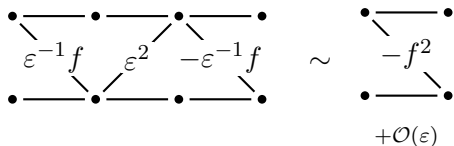
base



addition



squaring (idea)

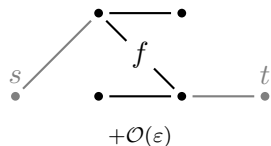


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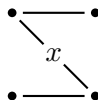
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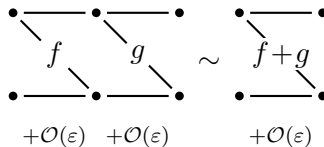
goal



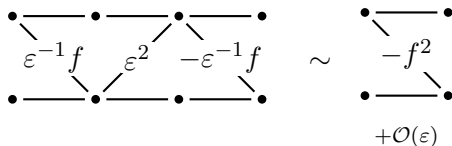
base



addition



squaring (idea)

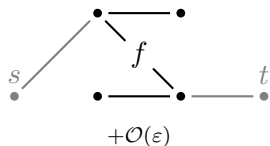


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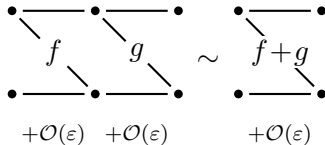
goal



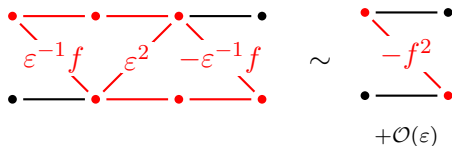
base



addition



squaring (idea)



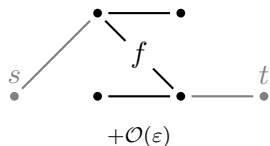
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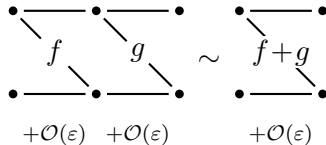
goal



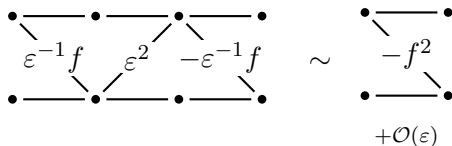
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squaring (idea)



multiplication  $fg = \frac{1}{2}((f+g)^2 - f^2 - g^2)$

# Statement of nondeterminism result

Recall:  $(g_n) \in \mathbf{VP}_1$  means  $g_n$  is product of  $\text{poly}(n)$  many affine linear forms

Definition:  $(f_n) \in \mathbf{VNP}_1$  if

- $\exists (g_n) \in \mathbf{VP}_1$
- $f_n(x_1, \dots, x_{p(n)}) = \sum_{b \in \{0,1\}^{\text{poly}(n)}} g_n(x_1, \dots, x_{p(n)}, b_1, \dots, b_{\text{poly}(n)})$

Naturally generalises to  $\mathbf{VNP}_e$  and  $\mathbf{VNP}$

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Valiant (1980):  $\mathbf{VNP}_e = \mathbf{VNP}$

Theorem:  $\mathbf{VNP}_1 = \mathbf{VNP}$

Corollary: strict inclusions  $\mathbf{VP}_1 \subsetneq \mathbf{VNP}_1$  and  $\mathbf{VP}_2 \subsetneq \mathbf{VNP}_2$

$$\overline{\mathbf{VP}_1} \subsetneq \overline{\mathbf{VP}_2} = \overline{\mathbf{VP}_e} \subseteq \overline{\mathbf{VP}}$$

$$\parallel \quad \cup \nparallel \quad \cup \perp \quad \cup \perp$$

$$\mathbf{VP}_1 \subsetneq \mathbf{VP}_2 \subsetneq \mathbf{VP}_e \subseteq \mathbf{VP}$$

$$\nparallel \cap \quad \nparallel \cap \quad \cap \cap \quad \cap \cap$$

$$\mathbf{VNP}_1 = \mathbf{VNP}_2 = \mathbf{VNP}_e = \mathbf{VNP}$$

$$\begin{array}{ccccccc}
 \overline{\mathbf{VP}_1} & \subsetneq & \overline{\mathbf{VP}_2} & = & \overline{\mathbf{VP}_e} & \subseteq & \overline{\mathbf{VP}} \\
 \parallel & & \cup \nsubseteq & & \cup \cap & & \cup \cap \\
 \mathbf{VP}_1 & \subsetneq & \mathbf{VP}_2 & \subsetneq & \mathbf{VP}_e & \subseteq & \mathbf{VP} \\
 \nsubseteq \cap & & \nsubseteq \cap & & \cap \cap & & \cap \cap \\
 \mathbf{VNP}_1 & = & \mathbf{VNP}_2 & = & \mathbf{VNP}_e & = & \mathbf{VNP}
 \end{array}$$

Thank you!

# Proof sketch $\mathbf{VNP}_1 = \mathbf{VNP}$

1. We know  $\mathbf{VP}_e \subseteq \mathbf{VP}_3$  (Ben-Or–Cleve).
2. We prove  $\mathbf{VP}_3 \subseteq \mathbf{VNP}_1$ . Construction: let nondeterminism select  $s$ - $t$  paths in width-3 ABP.
3. This shows  $\mathbf{VP}_e \subseteq \mathbf{VNP}_1$ . This implies  $\mathbf{VNP}_e \subseteq \mathbf{VNP}_1$ . We know  $\mathbf{VNP} = \mathbf{VNP}_e$  (Valiant).

## Side result: the continuant

Definition continuant

$$F_0 = 1$$

$$F_1(x_1) = x_1$$

$$F_n(x_1, \dots, x_n) = x_n \cdot F_{n-1}(x_1, \dots, x_{n-1}) \\ + F_{n-2}(x_1, \dots, x_{n-2})$$

Example:  $F_n(1, 1, \dots, 1) = n$ th Fibonacci number

Continuant complexity

$L_F(f) =$  smallest  $n$  such that  $f(x_1, \dots, x_n) = F_n(\ell_1, \dots, \ell_n)$

$L_F$  induces classes  $\mathbf{VP}_F$  and  $\overline{\mathbf{VP}_F}$

**Proposition:**  $\overline{\mathbf{VP}_F} = \overline{\mathbf{VP}_e}$

$$\begin{array}{cccccccccccc}
\overline{\text{VNP}}_1^{\text{wst}} & \not\subseteq & \overline{\text{VNP}}_1^{\text{w}} & \not\subseteq & \overline{\text{VNP}}_1^{\text{g}} & = & \overline{\text{VNP}}_2^{\text{wst}} & = & \overline{\text{VNP}}_2^{\text{w}} & = & \overline{\text{VNP}}_2^{\text{g}} & = & \overline{\text{VNP}}_e & = & \overline{\text{VNP}}_s & = & \overline{\text{VNP}} \\
\parallel & & \parallel & & \cup & \nearrow & \cup & \nearrow & \cup & \nearrow & \cup & \nearrow & \cup & \nearrow & \cup & \nearrow & \cup \\
\text{VNP}_1^{\text{wst}} & \not\subseteq & \text{VNP}_1^{\text{w}} & \not\subseteq & \text{VNP}_1^{\text{g}} & = & \text{VNP}_2^{\text{wst}} & = & \text{VNP}_2^{\text{w}} & = & \text{VNP}_2^{\text{g}} & = & \text{VNP}_e & = & \text{VNP}_s & = & \text{VNP} \\
\parallel & & \parallel & & \cup & \nearrow & \cup & \nearrow & \cup & \nearrow & \cup & \nearrow & \cup & \nearrow & \cup & \nearrow & \cup \\
\overline{\text{VP}}_1^{\text{wst}} & \not\subseteq & \overline{\text{VP}}_1^{\text{w}} & \not\subseteq & \overline{\text{VP}}_1^{\text{g}} & \not\subseteq & \overline{\text{VP}}_2^{\text{wst}} & = & \overline{\text{VP}}_2^{\text{w}} & = & \overline{\text{VP}}_2^{\text{g}} & = & \overline{\text{VP}}_e & \subseteq & \overline{\text{VP}}_s & \subseteq & \overline{\text{VP}} \\
\parallel & & \parallel & & \parallel & \nearrow & \cup & \nearrow & \cup & \nearrow & \cup & \nearrow & \cup & \nearrow & \cup & \nearrow & \cup \\
\overline{\text{VP}}_1^{\text{wst poly}} & \not\subseteq & \overline{\text{VP}}_1^{\text{w poly}} & \not\subseteq & \overline{\text{VP}}_1^{\text{g poly}} & \not\subseteq & \overline{\text{VP}}_2^{\text{wst poly}} & = & \overline{\text{VP}}_2^{\text{w poly}} & = & \overline{\text{VP}}_2^{\text{g poly}} & = & \overline{\text{VP}}_e^{\text{poly}} & \subseteq & \overline{\text{VP}}_s^{\text{poly}} & \subseteq & \overline{\text{VP}}^{\text{poly}} \\
\parallel & & \parallel & & \parallel & \cup & \cup & \cup & \cup & \cup & \cup & \cup & \parallel & \parallel & \parallel & \parallel & \parallel \\
\text{VP}_1^{\text{wst}} & \not\subseteq & \text{VP}_1^{\text{w}} & \not\subseteq & \text{VP}_1^{\text{g}} & \not\subseteq & \text{VP}_2^{\text{wst}} & \subseteq & \text{VP}_2^{\text{w}} & \not\subseteq & \text{VP}_2^{\text{g}} & \not\subseteq & \text{VP}_e & \subseteq & \text{VP}_s & \subseteq & \text{VP}
\end{array}$$