Noise Stability is Low-Dimensional

Anindya De, Elchanan Mossel, Joe Neeman

Gaussian noise stability

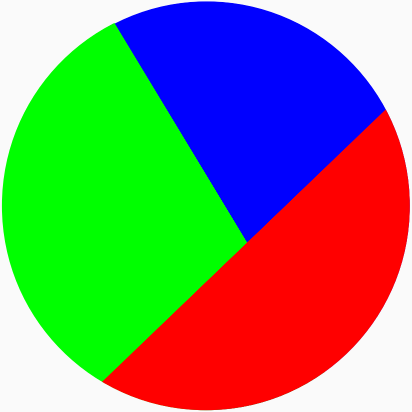
Take X and Y a pair of ρ-correlated Gaussians in $\mathbb{R}^n$ $(0 < \rho < 1)$.

For a partition $A = (A_1, \ldots, A_k)$ of $\mathbb{R}^n$ into k parts, define

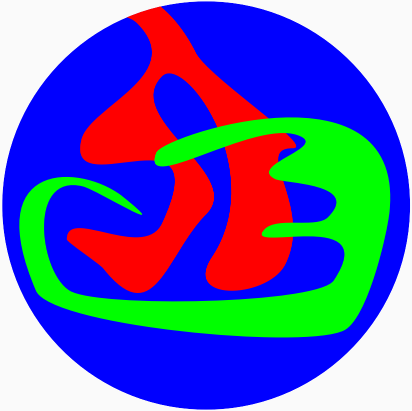
$$\mathrm{Stab}_\rho(A) = \Pr(X \text{ and } Y \text{ land in the same part})$$

Gaussian noise stability

Noise stable



Not noise stable



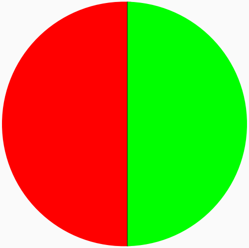
Gaussian noise stability

Theorem (Borell '85)

For a partition of $\mathbb{R}^n$ into two parts of equal Gaussian measure,

$$\mathrm{Stab}_\rho(A) \le \frac{1}{2} + \frac{\sin^{-1}\rho}{\pi}.$$

Equality is attained for a partition into half-spaces.



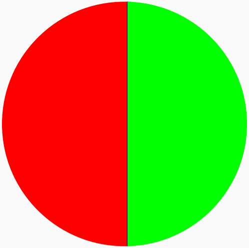
Gaussian noise stability

Theorem (Borell '85)

For a partition of $\mathbb{R}^n$ into two parts of equal Gaussian measure,

$$\mathrm{Stab}_\rho(A) \leq \frac{1}{2} + \frac{\sin^{-1}\rho}{\pi}.$$

Equality is attained for a partition into half-spaces.



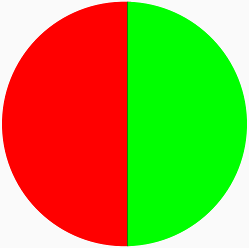
- well-known links to computational complexity (KKMO '05)

Gaussian noise stability

Theorem (Borell '85)
For a partition of $\mathbb{R}^n$ into two parts of equal Gaussian measure,

$$\mathrm{Stab}_\rho(A) \leq \frac{1}{2} + \frac{\sin^{-1}\rho}{\pi}.$$

Equality is attained for a partition into half-spaces.



- well-known links to computational complexity (KKMO '05)
- one-dimensional phenomenon

Gaussian noise stability

Theorem (???)

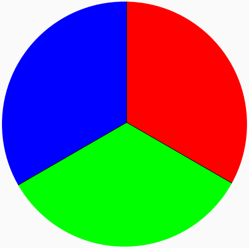
For a partition of $\mathbb{R}^n$ into three parts of equal Gaussian measure,

???

Gaussian noise stability

Conjecture (Peace sign conjecture)

The optimal partition looks like a peace sign.

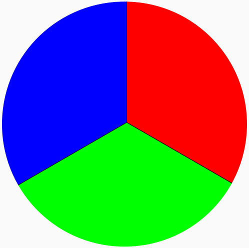


Gaussian noise stability

Conjecture (Peace sign conjecture)

The optimal partition looks like a peace sign.

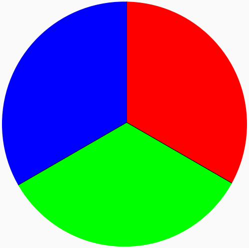
(two-dimensional phenomenon)



Gaussian noise stability

Conjecture (Peace sign conjecture)

The optimal partition looks like a peace sign.
(two-dimensional phenomenon)



Conjecture (Multi-dimensional peace sign conjecture)

For partitions into k equal measures, the optimal partition occurs in $\mathbb{R}^{k-1}$. It looks like a multi-dimensional peace sign.

Gaussian noise stability

Theorem (De-Mossel-N.)
For any k and any $\epsilon > 0$, there is a computable $n_0 = n_0(k, \epsilon)$ such that an ϵ-approximately optimal partition occurs in $\mathbb{R}^{n_0}$.

Gaussian noise stability

Theorem (De-Mossel-N.)

For any k and any $\epsilon > 0$, there is a computable $n_0 = n_0(k, \epsilon)$ such that an ϵ-approximately optimal partition occurs in $\mathbb{R}^{n_0}$.

Corollary

The optimal value of k-part noise stability is computable.

Gaussian noise stability

Theorem (De-Mossel-N.)

For any k and any $\epsilon > 0$, there is a computable $n_0 = n_0(k, \epsilon)$ such that an ϵ-approximately optimal partition occurs in $\mathbb{R}^{n_0}$.

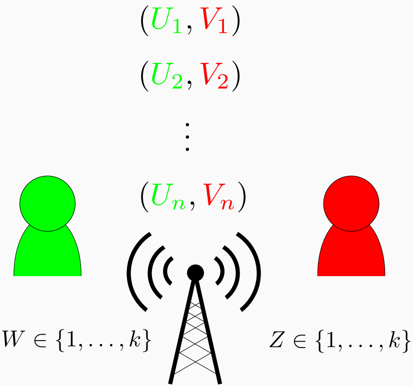
Corollary

The optimal value of k-part noise stability is computable.

Corollary (sort of)

The non-interactive correlation distillation value with k-ary outputs is computable.

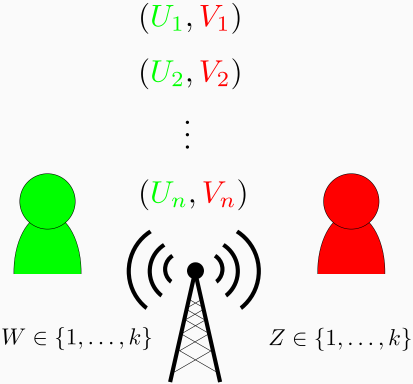
Correlation distillation



Goal: produce uniform output, agree with maximal probability.

What is the probability of agreement?

Correlation distillation



Goal: produce uniform output, agree with maximal probability.

What is the probability of agreement?

Ghazi-Kamath-Sudan '16: reduction to correlated Gaussian signals

The main theorem

Theorem (De-Mossel-N.)
For any k and any $\epsilon > 0$, there is a computable $n_0 = n_0(k, \epsilon)$ such that an ϵ-approximately optimal partition occurs in $\mathbb{R}^{n_0}$.

Proof outline

Idea: take an optimal partition in $\mathbb{R}^n$ (n huge) and try to "simulate" it in $\mathbb{R}^{n_0}$.

Proof outline

Idea: take an optimal partition in $\mathbb{R}^n$ (n huge) and try to "simulate" it in $\mathbb{R}^{n_0}$.

1. An optimal partition is close to a bounded-degree polynomial threshold function (PTF)

Proof outline

Idea: take an optimal partition in $\mathbb{R}^n$ (n huge) and try to "simulate" it in $\mathbb{R}^{n_0}$.

1. An optimal partition is close to a bounded-degree polynomial threshold function (PTF)
2. A bounded-degree PTF can be approximately simulated by a bounded-degree PTF on a bounded number of variables

Step 1: approximation by polynomials

Approximation by polynomials

Think of a partition as a function $f : \mathbb{R}^n \to \{e_1, \ldots, e_k\} \subset \mathbb{R}^k$.

Approximation by polynomials

Think of a partition as a function $f : \mathbb{R}^n \to \{e_1, \ldots, e_k\} \subset \mathbb{R}^k$.

Hermite expansion $f(x) = \sum_{\alpha,i} \hat{f}_{\alpha,i} H_\alpha(x) e_i$

Approximation by polynomials

Think of a partition as a function $f : \mathbb{R}^n \to \{e_1, \ldots, e_k\} \subset \mathbb{R}^k$.

Hermite expansion $f(x) = \sum_{\alpha,i} \hat{f}_{\alpha,i} H_\alpha(x) e_i$

Facts: $1 = \sum_{\alpha,i} \hat{f}^2_{\alpha,i}$ and $\mathrm{Stab}_\rho(f) = \sum_{\alpha,i} \rho^{\deg(H_\alpha)} \hat{f}^2_{\alpha,i}$

Approximation by polynomials

Think of a partition as a function $f : \mathbb{R}^n \to \{e_1, \ldots, e_k\} \subset \mathbb{R}^k$.

Hermite expansion $f(x) = \sum_{\alpha,i} \hat{f}_{\alpha,i} H_\alpha(x) e_i$

Facts: $1 = \sum_{\alpha,i} \hat{f}^2_{\alpha,i}$ and $\mathrm{Stab}_\rho(f) = \sum_{\alpha,i} \rho^{\deg(H_\alpha)} \hat{f}^2_{\alpha,i}$

Idea: noise stability $\Rightarrow$ lots of "low-degree" weight

$\Rightarrow$ approximate f by truncating the expansion

Approximation by polynomials

Think of a partition as a function $f : \mathbb{R}^n \to \{e_1, \ldots, e_k\} \subset \mathbb{R}^k$.

Hermite expansion $f(x) = \sum_{\alpha,i} \hat{f}_{\alpha,i} H_\alpha(x) e_i$

Facts: $1 = \sum_{\alpha,i} \hat{f}^2_{\alpha,i}$ and $\mathrm{Stab}_\rho(f) = \sum_{\alpha,i} \rho^{\deg(H_\alpha)} \hat{f}^2_{\alpha,i}$

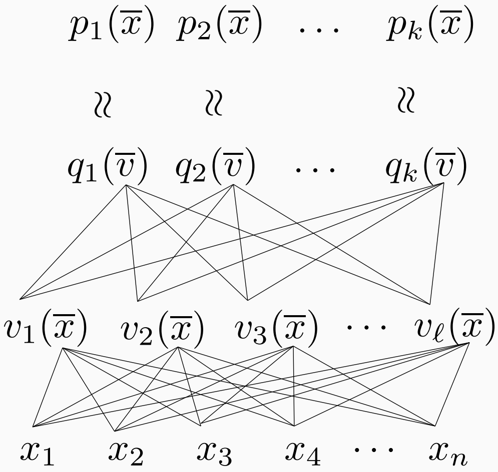
Idea: noise stability $\Rightarrow$ lots of "low-degree" weight

$\Rightarrow$ approximate f by truncating the expansion

Real proof goes through a smoothing/rounding procedure, and a connection with Gaussian surface area (KNOW '15).

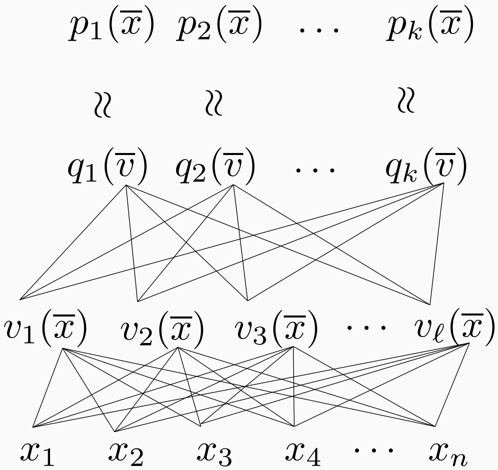
Step 2: dimension reduction

Polynomial structure theorem (De-Servedio)

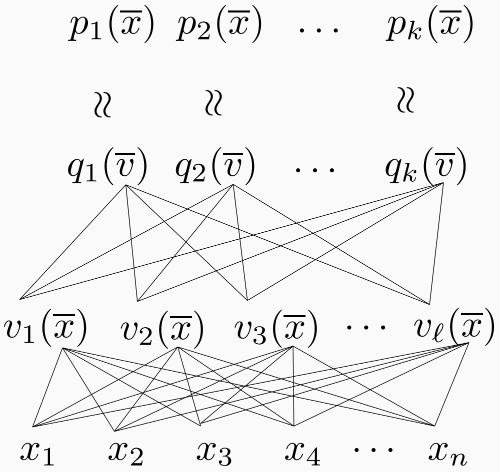


where ℓ is bounded and $v_1, \dots, v_\ell$ are "nice"

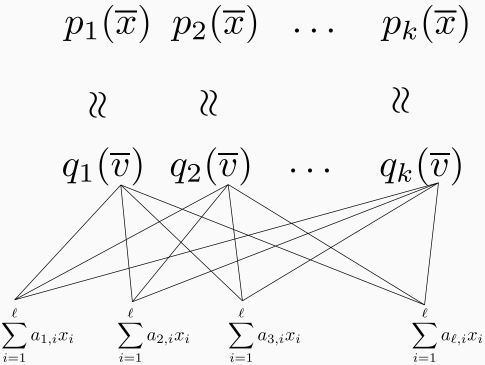
Polynomial Central Limit Theorem (De-Servedio)



Polynomial Central Limit Theorem (De-Servedio)



"Nice" polynomials satisfy a CLT, so they may as well be linear functions of ℓ variables

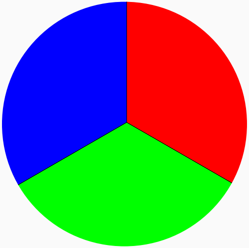


"Nice" polynomials satisfy a CLT, so they may as well be linear functions of ℓ variables

Open problem

Conjecture (Peace sign conjecture)

The optimal partition looks like a peace sign.



Thank you!