On the sum-of-squares degree of symmetric quadratic functions.

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Overview

1. Approximate sum-of-squares degree
2. ℓ_∞ approximate sos-degree
3. LRS bound tightness

Introduction

- Let $f : \{0,1\}^n \to \mathbb{R}_+$, then sos-deg$(f)$ is the minimum d such that $f(x) = \sum_{i \in [r]} h_i(x)^2$ for all $x \in \{0,1\}^n$ and deg$(h_i) \leq d$.

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- [Lee, Raghavendra, Steurer 15] in a breakthrough result showed that the psd-rank of S is $2^{\Omega(n^{2/11})}$.

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- We study ℓ_∞ and ℓ_1 approximate sos-deg of the functions f_k.

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  - We construct a quantum protocol showing that $QE_\epsilon(f)$ is $O(\sqrt{k(n-k)} + \sqrt{n \log(1/\epsilon)})$.
- Our running example to illustrate both these results will be $f_1(x) = (|x| - 1)(|x| - 2)$.

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- Provides simple proof of Grigoriev's sos-degree lower bound for $f_{\lfloor n/2 \rfloor}(x)$.

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  - $h_1(0) = 1$.
- Paturi's lemma: Let $p(x)$ be a univariate polynomial bounded by c on integer points $\{0, 1, \cdots, n\}$ that has a large derivative $|p'(\alpha)| \geq \delta$ at α. Then $\deg(p) = \Omega(\frac{\delta}{c}\sqrt{(\alpha(n - \alpha)}))$.

Upper bound

- Sampling algorithm: If $x_t = 1$ and i, j are sampled from the uniform distribution $U([n] \setminus t)$:

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- Our results imply that the above bound is tight and the LRS exponent can not be improved in a black-box manner.
- Proof idea: Construct polynomial $h(|x|) = f(|x|) + e(|x|)$ such that $h(|x|)$ is globally non-negative and error $\sum_i \binom{n}{i} |e(i)|$ is small.

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  - Find an explicit certificate for lower bound on sos-deg(f_k, ℓ_∞).