Identity Testing for constant-width, and commutative, ROABPs

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†supported by DST-SERB

Polynomial Identity Testing

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- Input Models:
  - Arithmetic Circuits
  - Arithmetic Branching Programs

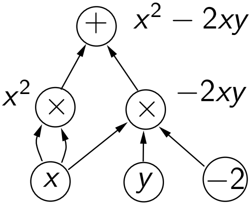

Figure : An Arithmetic circuit

Randomized Test

- Rephrasing the question: Given an arithmetic circuit decide if it computes the zero polynomial.
- Randomized PIT: evaluate $P(\mathbf{x})$ at a random point [Demillo and Lipton, 1978, Zippel, 1979, Schwartz, 1980].

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- Derandomizing PIT has connections with circuit lower bounds [Kabanets and Impagliazzo, 2003, Agrawal, 2005].
- An efficient test is known only for restricted classes of circuits, e.g., Sparse polynomials, set-multilinear circuits, ROABP.

Arithmetic Branching Programs

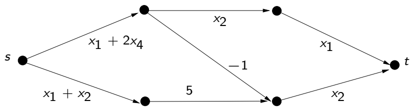

Figure : An Arithmetic branching program.

- ABP: a directed acyclic graph G with a start node and an end node.
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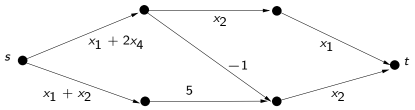

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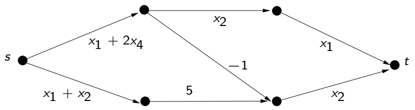

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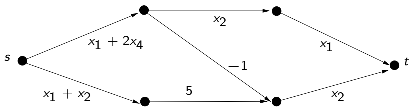

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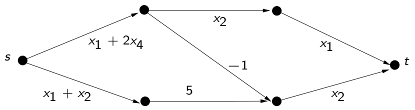

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- Equivalent representation:

$$\begin{bmatrix} x_1 + 2x_4 & x_1 + x_2 \end{bmatrix} \begin{bmatrix} x_2 & -1 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

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- Width-3 ABPs have the same expressive power as arithmetic formulas [Ben-Or and Cleve, 1992].
- Deterministic PIT: only for special ABPs, e.g. read-once oblivious ABP.

Read-once Oblivious ABP

- Any variable occurs in at most one layer.

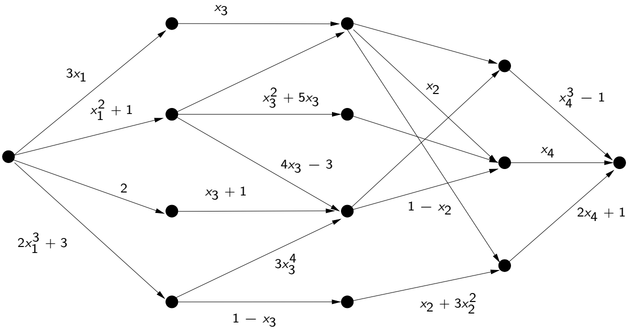

Figure : A Read-once oblivious ABP with variable order (x_1, x_3, x_2, x_4)

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- $d^{O(\log w)}(nw)^{O(\log \log w)}$-time blackbox test [Forbes et al., 2014]
  - for n variables, width w and individual degree d.
- We improve it to $(dnw)^{O(\log \log w)}$-time.

Read-once Ordered Branching Programs

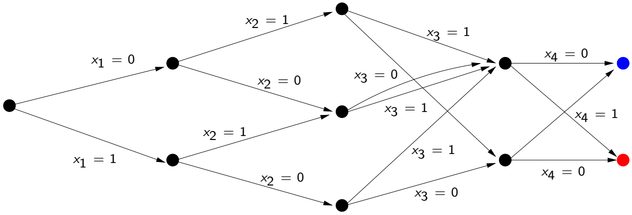

Figure : An ROBP

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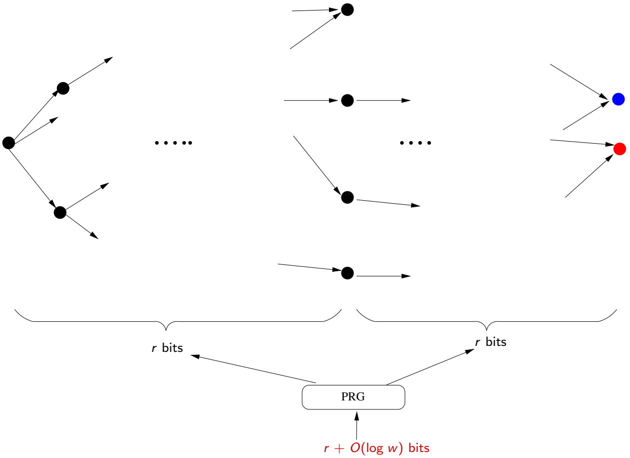

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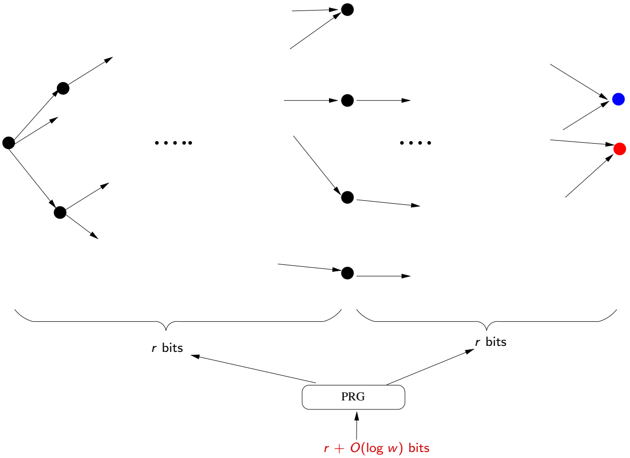
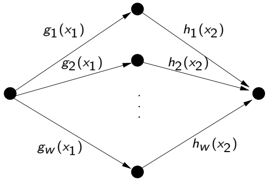


- Sample space size: poly$(w) \times 2^r$ instead of trivial $2^r \times 2^r$.

Hitting-set for Bivariate ROABP



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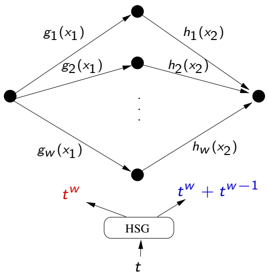

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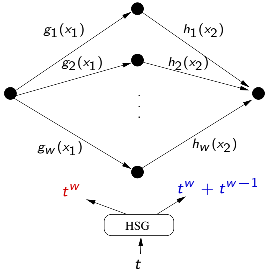

- $f(x_1, x_2) = \sum_{r=1}^{w} g_r(x_1)\ h_r(x_2)$
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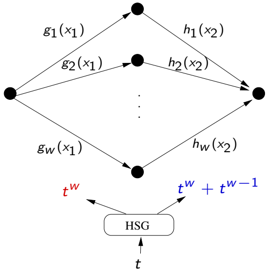

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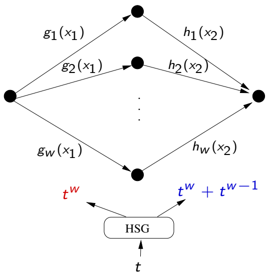

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- Hitting-set size: $2wd + 1$, instead of trivial $(d+1) \times (d+1)$.

n-variate ROABP

$$f = \begin{bmatrix} x_1 & & \end{bmatrix} \begin{bmatrix} & & \\ x_2 & & \\ & & \end{bmatrix} \begin{bmatrix} & & \\ x_3 & & \\ & & \end{bmatrix} \cdots \begin{bmatrix} & & \\ x_{n-1} & & \\ & & \end{bmatrix} \begin{bmatrix} \\ x_n \\ \\ \end{bmatrix}$$

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- $f({\color{red}t_1^w}, {\color{blue}t_1^w + t_1^{w-1}}) \neq 0$ (bivariate ROABP).

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- Known variable order.

Proof of the bivariate case

- *Claim:* If $f(x,y) = \sum_{r=1}^{w} g_r(x)h_r(y)$, then $f(t^w, t^w + t^{w-1}) \neq 0$.

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- Coefficient Matrix for $f(x,y)$ [Nisan, 1991]

$$
\begin{array}{c c}
 & \begin{array}{ccccc} y^0 & \cdots & y^j & \cdots & y^d \end{array} \\
\begin{array}{c} x^0 \\ \vdots \\ x^i \\ \vdots \\ x^d \end{array} &
\begin{bmatrix} & & & & \\ & & & & \\ - & & \mathrm{coef}_f(x^i y^j) & & \\ & & & & \\ & & & & \end{bmatrix}
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- Define rank(f) as the rank of this matrix.
- *Claim:* $\mathrm{rank}(f) \leq w$ [Nisan, 1991].

Proof of the bivariate case

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- *Claim:* $\mathrm{rank}(f_r) \leq 1$.

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$$\begin{array}{cc} & \begin{array}{cccc} y^0 & y^1 & \cdots & y^d \end{array} \\ \begin{array}{c} x^0 \\ x^1 \\ \vdots \\ x^d \end{array} & \begin{bmatrix} a_0 b_0 & a_0 b_1 & & a_0 b_d \\ a_1 b_0 & a_1 b_1 & & a_1 b_d \\ \vdots & \vdots & \cdots & \vdots \\ a_d b_0 & a_d b_1 & & a_d b_d \end{bmatrix} \end{array}$$

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\begin{bmatrix}
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a_1 b_0 & a_1 b_1 & & a_1 b_d \\
\vdots & \vdots & \cdots & \vdots \\
a_d b_0 & a_d b_1 & & a_d b_d
\end{bmatrix}
\end{array}
$$

- $\implies \text{rank}(f) = \text{rank}(\sum_{r=1}^{w} f_r) \leq w$.

Proof of the bivariate case

$$(x, y) \mapsto (t^w, t^w + t^{w-1}) = (t^w, t^w(1 + t^{-1})).$$

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Proof of the bivariate case

$$(x,y) \mapsto ({\color{red}t^w}, {\color{blue}t^w + t^{w-1}}) = ({\color{red}t^w}, {\color{blue}t^w(1+t^{-1})}).$$

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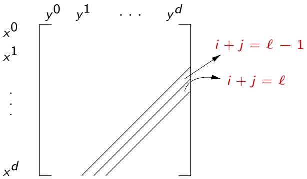
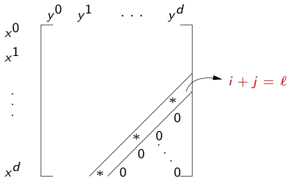
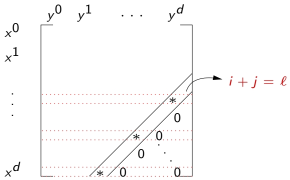


Proof of the bivariate case

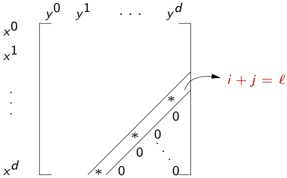


Proof of the bivariate case



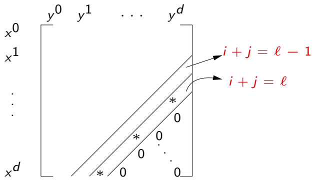
- Leading nonzero Diagonal: at most w nonzero entries.

Proof of the bivariate case



- Leading nonzero Diagonal: at most w nonzero entries.
- Leading term: $t^{w\ell}$.

Proof of the bivariate case



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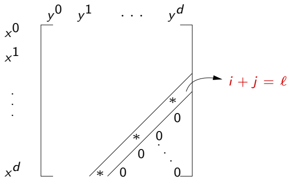

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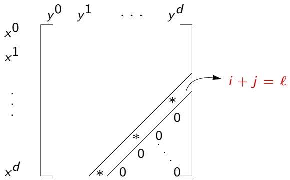

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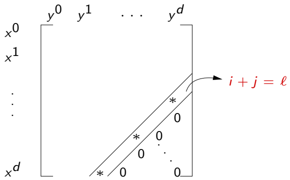

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- ℓ-th diagonal nonzero monomials: $\{x^{\ell-j_1}y^{j_1}, x^{\ell-j_2}y^{j_2}, \cdots, x^{\ell-j_w}y^{j_w}\}$.

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$$(x, y) \mapsto (t^w, t^w(1 + t^{-1})).$$
$$x^{\ell - j_1} y^{j_1} \mapsto t^{\ell w}(1 + t^{-1})^{j_1}.$$

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\begin{aligned}
x^{\ell-j_1}y^{j_1} &\mapsto \quad \binom{j_1}{0}t^{\ell w} + \binom{j_1}{1}t^{\ell w-1} + \cdots + \binom{j_1}{w-1}t^{(\ell-1)w+1} + \cdots \\
x^{\ell-j_2}y^{j_2} &\mapsto \quad \binom{j_2}{0}t^{\ell w} + \binom{j_2}{1}t^{\ell w-1} + \cdots + \binom{j_2}{w-1}t^{(\ell-1)w+1} + \cdots \\
&\qquad\qquad\qquad\qquad \vdots \\
x^{\ell-j_w}y^{j_w} &\mapsto \quad \binom{j_w}{0}t^{\ell w} + \binom{j_w}{1}t^{\ell w-1} + \cdots + \binom{j_w}{w-1}t^{(\ell-1)w+1} + \cdots
\end{aligned}
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\begin{array}{rcl}
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 & & \vdots \\
x^{\ell-j_w}y^{j_w} & \mapsto & \binom{j_w}{0}t^{\ell w} + \binom{j_w}{1}t^{\ell w-1} + \cdots + \binom{j_w}{w-1}t^{(\ell-1)w+1} + \cdots \\
\hline
 & & 0 \qquad * \qquad \cdots \qquad 0
\end{array}
$$

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 & & \vdots \\
x^{\ell-j_w}y^{j_w} & \mapsto & \binom{j_w}{0}t^{\ell w} + \binom{j_w}{1}t^{\ell w-1} + \cdots + \binom{j_w}{w-1}t^{(\ell-1)w+1} + \cdots \\
\hline
 & & 0 \qquad * \qquad \cdots \qquad 0
\end{array}
$$

- Assuming $j_k \neq j_{k'}$ requires nonzero characteristic.

Discussion

- Possible improvements:
  - Unknown variable order
  - Hitting-set for all fields.
  - Poly-time for arbitrary width.

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- Connections between arithmetic and boolean pseudorandomness?

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