A Composition Theorem for Conical Juntas

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AND-OR trees

1	Set-disjointness $\text{OR}_n \circ \text{AND}_2$	$\Omega(n)$	[KS'87] [Raz'91] [BJKS'04] **(info)**
2	Tribes $\text{AND}_{\sqrt{n}} \circ \text{OR}_{\sqrt{n}} \circ \text{AND}_2$	$\Omega(n)$	[JKS'03] **(info)** [HJ'13]
k	$\text{AND}_{n^{1/k}} \circ \cdots \circ \text{OR}_{n^{1/k}} \circ \text{AND}_2$	$n/2^{O(k)}$	[JKR'09] **(info)** [LS'10] **(info)**

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log *n*	$\text{AND}_2 \circ \cdots \circ \text{OR}_2 \circ \text{AND}_2$	$O(n^{0.753...})$ $\Omega(\sqrt{n})$	[Snir'85] [JKZ'10]

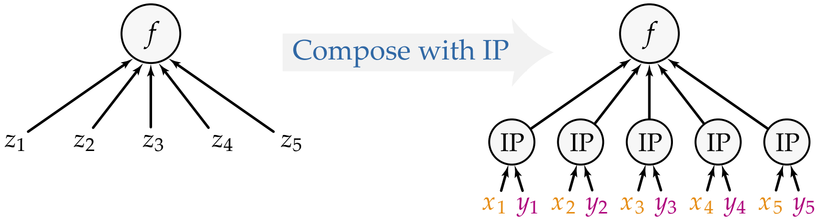
↳ Gap!

New tool – Conical juntas

Communication-to-query theorem [GLMWZ'15]:

For every boolean function $f\colon \{0,1\}^n \to \{0,1\}$,

$$\mathbf{BPP}^{\mathbf{cc}}_{\epsilon}(f \circ \mathrm{IP}_{\log n}) \geq \Omega(\deg^{+}_{\epsilon}(f))$$



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- **Conical juntas:** Nonnegative combination of conjunctions

$$\textsc{Or}_2 : \quad \tfrac{1}{2}x_1 + \tfrac{1}{2}x_2 + \tfrac{1}{2}\bar{x}_1 x_2 + \tfrac{1}{2}x_1\bar{x}_2$$

- Approximate conical junta degree $\deg^{+}_{\epsilon}(f)$ is the least degree of a conical junta h such that

$$\forall x \in \{0,1\}^n : \quad |f(x) - h(x)| \leq \epsilon.$$

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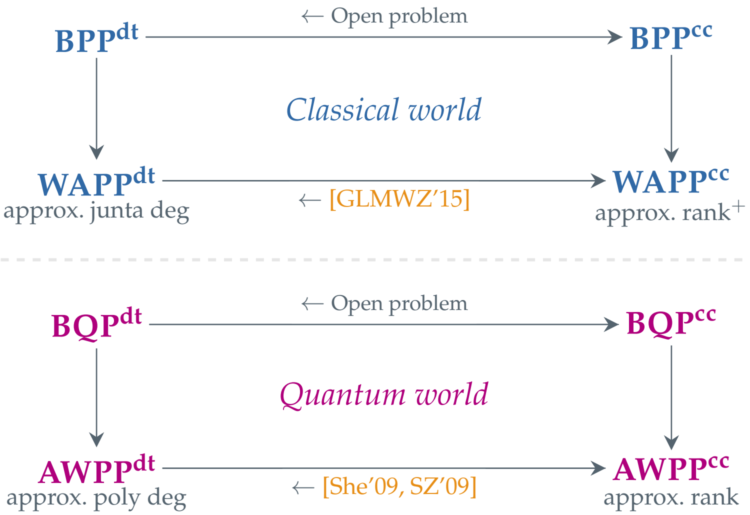
- **Previous talk:** 0-1 coefficients = Unambiguous DNFs

New tool – Big picture

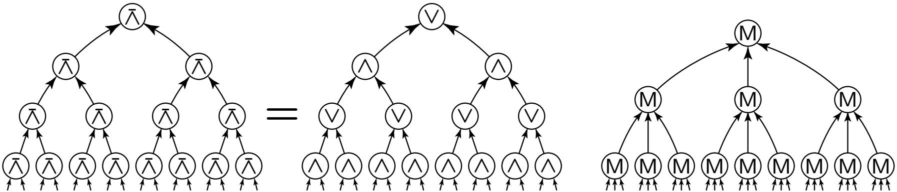


This work:

A Composition Theorem for Conical Juntas

That is: Want to understand approximate conical junta degree of $f \circ g$ in terms of f and g

Our results – Applications



$\text{NAND}^{\circ 4}$ $=$ $(\text{OR} \circ \text{AND})^{\circ 2}$ $\quad$ $\text{MAJ}_3^{\circ 3}$

Query:

- $\deg^{+}_{1/n}(\text{NAND}^{\circ k}) \geq \Omega(n^{0.753\ldots})$
- $\deg^{+}_{1/n}(\text{MAJ}_3^{\circ k}) \geq \Omega(2.59\ldots^{k})$

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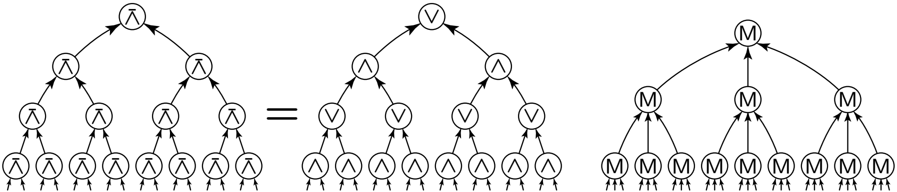

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Previously: $O(2.65^k) \geq \mathbf{BPP^{dt}}(\text{MAJ}_3^{\circ k}) \geq \Omega(2.57^k)$
[JKS'03, LNPV'06, Leo'13, MNSSTX'15]

Note: Unamplifiability of error

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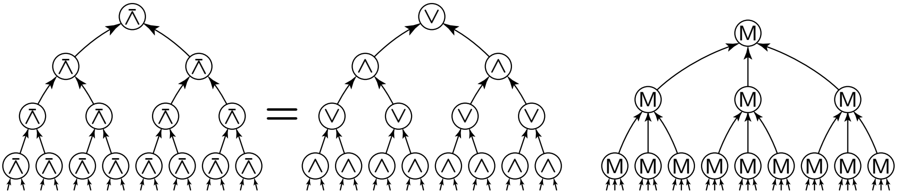

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Communication:
- $\mathbf{BPP^{cc}}(\textsc{Nand}^{\circ k}) \geq \tilde{\Omega}(n^{0.753...})$
- $\mathbf{BPP^{cc}}(\textsc{Maj}_3^{\circ k}) \geq \Omega(2.59^{k})$

Note: Log-factor loss

Formalising the composition theorem

Average degree

For $h = \sum w_C C$,

$$\begin{aligned} \text{adeg}_x(h) &:= \sum w_C |C| \cdot C(x) \\ \text{adeg}(h) &:= \max_x \ \text{adeg}_x(h) \end{aligned}$$

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First formalisation attempt

$$\mathrm{adeg}(f \circ g) \geq \mathrm{adeg}(f) \cdot \min\{\mathrm{adeg}(g), \mathrm{adeg}(\neg g)\} \quad \textbf{?}$$

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Counter-example! $\text{adeg}(\text{OR}_2 \circ \text{MAJ}_3) = 3.92... < 4$

$\mathrm{adeg}(h; b_0, b_1)$ – charge b_i for reading an input bit that is i

Primal / Dual for average degree of f

$$\begin{aligned} \min \quad & \mathrm{adeg}\left(\textstyle\sum w_C C; b_0, b_1\right) & \\ \textit{subject to} \quad & \textstyle\sum w_C C(x) = f(x), & \forall x \\ & w_C \geq 0, & \forall C \end{aligned}$$

$$\begin{aligned} \max \quad & \langle \Psi, f \rangle & \\ \textit{subject to} \quad & \langle \Psi, C \rangle \leq \mathrm{adeg}(C; b_0, b_1), & \forall C \\ & \Psi(x) \in \mathbb{R}, & \forall x \end{aligned}$$

Formalisation – Statement of theorem

Regular certificates: Circumventing the counter-example

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Composition Theorem

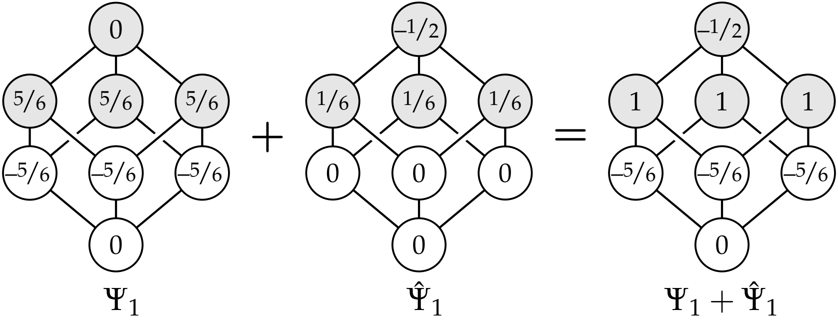
Suppose we have regular LP certificates witnessing

$$\begin{aligned} \text{adeg}(g) &\geq b_1 & \qquad \text{adeg}(f; b_0, b_1) &\geq a_1 \\ \text{adeg}(\neg g) &\geq b_0 & \qquad \text{adeg}(\neg f; b_0, b_1) &\geq a_0 \end{aligned}$$

then $f \circ g$ admits a regular LP certificate witnessing

$$\begin{aligned} \text{adeg}(f \circ g) &\geq a_1 \\ \text{adeg}(\neg f \circ g) &\geq a_0 \end{aligned}$$

Regular certificates for MAJ_3



	MAJ_3	$\text{MAJ}_3^{\circ 2}$	$\text{MAJ}_3^{\circ 3}$	$\text{MAJ}_3^{\circ 4}$
# dual variables:	3	5	9	17
lower bound:	2.5	$2.581\ldots^2$	$2.596\ldots^3$	*Open!*

Subsequent application

Eight-author paper: Anshu, Belovs, Ben-David, Göös, Jain, Kothari, Lee, and Santha [ECCC'16]

1. $\exists$ total F : $\mathbf{BPP^{cc}}(F) \geq \tilde{\Omega}(\mathbf{BQP^{cc}}(F)^{2.5})$
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Proof idea for 1

$$\deg^{+}_{\epsilon}(\textsc{Simon}_n \circ \textsc{And}_n \circ \textsc{Or}_n) \geq \Omega(n^{2.5})$$

$\Downarrow$ Communication-to-query

$$\mathbf{BPP^{cc}}(\textsc{Simon}_n \circ \textsc{And}_n \circ \textsc{Or}_n \circ \mathrm{IP}_{\log n}) \geq \tilde{\Omega}(n^{2.5})$$

$\Downarrow$ Cheat sheet technique

$$\mathbf{BPP^{cc}}\big((\textsc{Simon}_n \circ \textsc{And}_n \circ \textsc{Or}_n \circ \mathrm{IP}_{\log n})_{\mathrm{CS}}\big) \geq \tilde{\Omega}(n^{2.5})$$

Open problems

Composition theorems:

- Explain why our composition theorem works!-)
- Better certificates for $\textsc{Maj}_3^{\circ k}$?
- Does a composition theorem hold for $\mathbf{BPP}^{\mathbf{dt}}$?

Simulation theorems:

- Communication-to-query simulation for **BPP**?
- Constant-size gadgets for junta-based simulation?
- More things to do with conical juntas?

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Cheers!